MOTT-RUTHERFORD SCATTERING AND BEYOND

Abstract

The electron charge is considered to be distributed or extended in space. The differential of the electron charge is set equal to a function of the electron charge coordinates multiplied by a four-dimensional differential volume element. The four-dimensional integral of this function is required to equal the electron charge in all Lorentz frames. The S-matrix for the scattering of such an electron by a Coulomb potential is calculated. This modification is related to replacing the classical potential energy of a point electron in a Coulomb potential by the classical potential energy of an extended electron in a Coulomb potential. The result is that the S-matrix of the extended electron theory is a product of the S-matrix of the point electron theory times an expression which is dependent on the electron size and structure and is called the electron form factor.

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I. INTRODUCTION

Lorentz theory associates a finite radius with the electron. This removes singularities on the world line of the electron. However, the theory is not relativistically invariant. Attempts to remove the singularities associated with a point electron have been discussed by Schweber.[1] Among those attempts was the work of McManus. He constructed a Lorentz invariant classical theory of the electron with an extended charge distribution.[2] A function of the invariant distance between two spacetime points was introduced in the integral for the electric current. The theory was later extended to quantum electrodynamics(QED), specifically, to calculations of the self-mass and the vacuum polarization.[3]

As mentioned in the Abstract, the differential of the electron charge is set equal to a function multiplied by a four-dimensional differential volume element. In the next section, that function is determined by the requirement that the four-dimensional integral of the function is equal to the electron charge in all Lorentz frames. The third section reviews the calculation of the S-matrix for point electron scattering by a Coulomb potential. In the fourth section, the S-matrix for electron scattering by a Coulomb potential is again calculated, but with the electron charge replaced by a four dimensional integral. In addition,

the Coulomb potential must be evaluated at the charge points. It is shown that the new S-matrix is the product of the S-matrix of the point electron theory times a term dependent on the size and structure of the electron, which is called the electron form factor. At low speeds, the electron size term is approximately one, so that the point electron theory and the extended electron theory are not experimentally distinguishable. Differences between the theories occur at high electron speeds. The fifth section calculates the form factor for a specific electron charge distribution. The sixth section treats scattering with an alternative electron charge density and is followed by a short discussion.

II. EXTENDED ELECTRON CHARGE

In the rest frame of the electron, let $x_r'^{\mu} = (x_r'^0, x_r'^1, x_r'^2, x_r'^3)$ denote a spacetime charge point, and let $x_r^{\mu} = (x_r^0, x_r^1, x_r^2, x_r^3)$ denote the center of the electron charge distribution. The charge distribution of the electron is assumed to have a well-defined center, which is identified as the argument of the wave function. The shape of the charge distribution depends on the motion of the charge, and is assumed to be unaffected by any interaction. Sometimes the superscript will be omitted, and we

will write $x'_r = (x'^0_r, x'^1_r, x'^2_r, x'^3_r)$ and $x_r = (x^0_r, x^1_r, x^2_r, x^3_r)$. Introduce $\tilde{x}_r = x'_r - x_r$ or equivalently $\tilde{x}^\mu_r = x'^\mu_r - x^\mu_r$. In a frame of reference in which the electron moves with a speed β_i in the $+x^3$ direction, let $x'_m = (x'^0_m, x'^1_m, x'^2_m, x'^3_m)$ denote a spacetime charge point, and let $x^\mu_m = (x^0_m, x^1_m, x^2_m, x^3_m)$ denote the center of the charge distribution. Introduce $\tilde{x}_m = x'_m - x_m$. Note that the m frame is what is usually called the "lab" frame. A Lorentz transformation yields $\tilde{x}^1_r = \tilde{x}^1_m$, $\tilde{x}^2_r = \tilde{x}^2_m$, $\tilde{x}^3_r = \gamma_i(\tilde{x}^3_m - \beta_i\tilde{x}^0_m)$, and $\tilde{x}^0_r = \gamma_i(\tilde{x}^0_m - \beta_i\tilde{x}^3_m)$ where $\gamma_i = 1/\sqrt{1-\beta_i^2}$. Denote this Lorentz transformation by $\tilde{x}^\mu_r = L(\tilde{x}^\mu_m)$. The inverse Lorentz transformation is $\tilde{x}^1_m = \tilde{x}^1_r$, $\tilde{x}^2_m = \tilde{x}^2_r$, $\tilde{x}^3_m = \gamma_i(\tilde{x}^3_r + \beta_i\tilde{x}^0_r)$, and $\tilde{x}^0_m = \gamma_i(\tilde{x}^0_r + \beta_i\tilde{x}^3_r)$.

In this paper, the word "invariant" is applied to a quantity which is unchanged by a Lorentz transformation. In particular, the numerical value of an invariant quantity is the same in all Lorentz frames. Examples are the electron charge and the speed of light. The scalar products of two four-vectors are equal in two arbitrary Lorentz frames, therefore the scalar product is an invariant. A specific example is $x_r \cdot x_r = x_m \cdot x_m$. To establish this equality, substitute $x_r = L(x_m)$ on the left hand side.

In the rest frame of the electron, the volume integral of the charge density $\rho_r(\tilde{x}_r)$ yields the electron charge $e < 0$, i.e., $\int \rho_r(\tilde{x}_r) d^3\tilde{x}_r = e$. Since $\int \delta(\tilde{x}^0_r) d\tilde{x}^0_r = 1$, $\int \rho_r(\tilde{x}_r)\delta(\tilde{x}^0_r) d^4\tilde{x}_r = e$ where δ denotes the delta

function.[4] Thus an element of charge in the rest frame, call it de_r, can be replaced by $\rho_r(\tilde{x}_r)\delta(\tilde{x}_r^0)d^4\tilde{x}_r$.

Recall that the zeroth component of J, the current density four-vector, equals the speed of light c times the charge density ρ. For convenience, set c equal to 1. Then $J^\mu = (\rho, J^1, J^2, J^3)$, so in the rest frame, $J_r^\mu = (\rho_r, 0, 0, 0)$. The unit step function $H(x)$ is defined by $H(x) = 1$ for $x > 0$ and $H(x) = 0$ for $x < 0$. The derivative of the unit step function is the delta function $\delta(x)$. In the rest frame, $J \cdot \nabla$ takes the form

$$J_r \cdot \nabla_r = J_r^0(\tilde{x}_r)\frac{\partial}{\partial \tilde{x}_r^0} \tag{1}$$

where ∇ is the four-dimensional gradient operator. So

$$e = \int \rho_r(\tilde{x}_r)\delta(\tilde{x}_r^0)d^4\tilde{x}_r = \int J_r^0(\tilde{x}_r)\frac{\partial H(\tilde{x}_r^0)}{\partial \tilde{x}_r^0}d^4\tilde{x}_r = \int (J_r \cdot \nabla_r)H(\tilde{x}_r^0)d^4\tilde{x}_r. \tag{2}$$

Let f be a function of $\tilde{x}_r^\mu$, Note that $f(\tilde{x}_r^\mu) = f(L(\tilde{x}_m^\mu))$, or in terms of components $f(\tilde{x}_r^0, \tilde{x}_r^1, \tilde{x}_r^2, \tilde{x}_r^3) = f(\gamma_i(\tilde{x}_m^0 - \beta_i\tilde{x}_m^3), \tilde{x}_m^1, \tilde{x}_m^2, \gamma_i(\tilde{x}_m^3 - \beta_i\tilde{x}_m^0)$. Since the Jacobian of a Lorentz transformation is one,

$$\int f(\tilde{x}_r)\, d^4\tilde{x}_r = \int f(L(\tilde{x}_m))\, d^4\tilde{x}_m\,.^5 \tag{3}$$

In the m frame, $J_m^\mu = (\rho_m, 0, 0, J_m^3)$, so $J \cdot \nabla$ takes the form

$$(J_m \cdot \nabla_m) = J^0_m(\tilde{x}_m)\frac{\partial}{\partial \tilde{x}^0_m} + J^3_m(\tilde{x}_m)\frac{\partial}{\partial \tilde{x}^3_m}. \tag{4}$$

The scalar product $J \cdot \nabla$ is invariant. In particular, $J_r \cdot \nabla_r = J_m \cdot \nabla_m$. This equality can be established by Lorentz transforming J_r and ∇_r on the left. By Eq (3) and invariance of the scalar product $J \cdot \nabla$,

$$e = \int (J_r \cdot \nabla_r) f(\tilde{x}_r) d^4\tilde{x}_r = \int (J_m \cdot \nabla_m) f(L(\tilde{x}_m))\, d^4\tilde{x}_m. \tag{5}$$

Let $f(\tilde{x}_r) = H(\tilde{x}^0_r)$, and find

$$e = \int (J_r \cdot \nabla_r) H(\tilde{x}_r) d^4\tilde{x}_r = \int (J_m \cdot \nabla_m) H[\gamma_i(\tilde{x}^0_m - \beta_i \tilde{x}^3_m)] d^4\tilde{x}_m.{}^4 \tag{6}$$

By the Lorentz transformation, $J^0_m(\tilde{x}_m) = \gamma_i \rho_r(\tilde{x}_r) = \gamma_i \rho_r(L(\tilde{x}_m))$, and $J^3_m(\tilde{x}_m) = \gamma_i \beta_i \rho_r(\tilde{x}_r) = \gamma_i \beta_i \rho_r(L(\tilde{x}_m))$. Then by Eq (6)

$$e = \int \rho_r(\tilde{x}_r)\delta(\tilde{x}^0_r) d^4\tilde{x}_r =$$

$$\int \Big[\gamma_i \rho_r(L(\tilde{x}_m))\gamma_i + \gamma_i \beta_i \rho_r(L(\tilde{x}_m))(-\gamma_i \beta_i)\Big] \delta[\gamma_i(\tilde{x}^0_m - \beta_i \tilde{x}^3_m)] d^4\tilde{x}_m =$$

$$\int \rho_r(L(\tilde{x}_m))\delta[\gamma_i(\tilde{x}^0_m - \beta_i \tilde{x}^3_m)] d^4\tilde{x}_m. \tag{7}$$

Thus, de_m, an element of charge in the m frame, can be set equal to $\rho_r(L(\tilde{x}_m))\delta[\gamma_i(\tilde{x}^0_m - \beta_i\tilde{x}^3_m)]d^4\tilde{x}_m$. Notice Eq (7) is in agreement with Eq (3).

We will now work with a specific electron charge distribution to illustrate the preceding equations. Consider a uniform spherical surface charge distribution at rest, which is centered at $\mathbf{x}_r = (x^1_r, x^2_r, x^3_r)$ and has a radius a. Recall x' is a spacetime charge point. Then,

$$\rho_r(x'_r) = \frac{e}{4\pi a^2}\delta(\sqrt{(x'^1_r - x^1_r)^2 + (x'^2_r - x^2_r)^2 + (x'^3_r - x^3_r)^2} - a) =$$

$$\rho_r(\tilde{x}_r) = \frac{e}{4\pi a^2}\delta(\sqrt{(\tilde{x}^1_r)^2 + (\tilde{x}^2_r)^2 + (\tilde{x}^3_r)^2} - a). \quad (8)$$

In spherical coordinates, let $(\tilde{r}_r)^2 = (\tilde{x}^1_r)^2 + (\tilde{x}^2_r)^2 + (\tilde{x}^3_r)^2$, so that $d^3\tilde{x}_r = (\tilde{r}_r)^2 \sin\tilde{\theta}_r\, d\tilde{\theta}_r\, d\tilde{\phi}_r\, d\tilde{r}_r$. Then,

$$\int \rho_r(\tilde{x}_r)\delta(\tilde{x}^0_r)d^4\tilde{x}_r = \int \frac{e\delta(\tilde{r}_r - a)(\tilde{r}_r)^2 \sin\tilde{\theta}_r\, d\tilde{\theta}_r\, d\tilde{\phi}_r\, d\tilde{r}_r}{4\pi a^2}\delta(\tilde{x}^0_r)d\tilde{x}^0_r =$$

$$\int \frac{e\delta(\tilde{r}_r - a)(\tilde{r}_r)^2 \sin\tilde{\theta}_r\, d\tilde{\theta}_r\, d\tilde{\phi}_r\, d\tilde{r}_r}{4\pi a^2} = e. \quad (9)$$

For the charge distribution given by Eq. (8),

$$\rho_r(L(\tilde{x}_m)) = \frac{e}{4\pi a^2}\,\delta(\sqrt{(\tilde{x}^1_m)^2 + (\tilde{x}^2_m)^2 + \gamma_i^2(\tilde{x}^3_m - \beta_i\tilde{x}^0_m)^2} - a). \quad (10)$$

It will be verified that Eq (7) is correct for the above charge distribution. The integral in Eq (7) can be written

$$\int \rho_r(L(\tilde{x}_m))\delta[\gamma_i(\tilde{x}^0_m - \beta_i\,\tilde{x}^3_m)]d^4\tilde{x}_m = \int \frac{e\delta(\sqrt{(\tilde{x}^1_m)^2 + (\tilde{x}^2_m)^2 + (\tilde{x}^3_m)^2/\gamma_i^2} - a)}{4\pi a^2}\frac{d^3\tilde{x}_m}{\gamma_i} \quad (11)$$

where $\tilde{x}^1_m, = \tilde{x}^1_r$, $\tilde{x}^2_m = \tilde{x}^2_r$, and $\tilde{x}^3_m/\gamma_i = \gamma_i(\tilde{x}^3_m - \beta_i\tilde{x}^0_m) = \tilde{x}^3_r$. Change to spherical coordinates where $(\tilde{r}_r)^2 = (\tilde{x}^1_r)^2 + (\tilde{x}^2_r)^2 + (\tilde{x}^3_r)^2$, The volume element is $d^3\tilde{x}_m/\gamma_i = d^3\tilde{x}_r = (\tilde{r}_r)^2\sin\hat{\theta}_r\,d\hat{\theta}_r\,d\hat{\phi}_r\,d\hat{r}_r$. Finally,

$$\int \rho_r(L(\tilde{x}_m))\delta[\gamma_i(\tilde{x}^0_m - \beta_i\,\tilde{x}^3_m)]d^4\tilde{x}_m = \int \frac{e\delta(\tilde{r}_r - a)(\tilde{r}_r)^2\sin\tilde{\theta}_r\,d\tilde{\theta}_r\,d\tilde{\phi}_r\,d\tilde{r}_r}{4\pi a^2} = e. \quad (12)$$

III. MOTT SCATTERING

The calculation of the S-matrix for the scattering of a moving point electron from a fixed Coulomb potential $A_0(\mathbf{x})$ will follow Bjorken and Drell.[6] For the point electron, the approximate S-matrix element is

$$S_{fi} = \int d^4x\,\bar{\phi}_f(x)(-ie\gamma^0)A_0(\mathbf{x})\phi_i(x). \quad (13)$$

The exact initial wave function is approximated by the plane wave solution to the Dirac equation. This plane wave solution, which is normalized to unity in a box of volume V, is

$$\phi_i(x) = \sqrt{\frac{m}{E_i V}}\, u(p_i, s_i) \exp\left(-ip_i \cdot x\right) \tag{14}$$

where $\hbar$ and c are set equal to 1, m is the electron rest mass, s_i is the initial spin, $p_i^\mu = (p_i^0 = E_i, p_i^1, p_i^2, p_i^3)$ is the initial four-momentum, $u(p_i, s_i)$ is a four-component spinor, and the four by four Dirac matrix $\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Each entry in γ^0 is actually a two by two matrix. Similarly, the final electron wave function is

$$\phi_f(x) = \sqrt{\frac{m}{E_f V}}\, u(p_f, s_f) \exp\left(-ip_f \cdot x\right) \tag{15}$$

where $p_f = (p_f^0 = E_f, p_f^1, p_f^2, p_f^3)$ is the final momentum, s_f is the final spin, $u(p_f, s_f)$ is a four component spinor, and $\bar{\phi}_f = \phi_f^\dagger \gamma^0$. For a point charge $-Ze > 0$, the Coulomb potential is $A_0(\mathbf{x}) = -Ze/4\pi|\mathbf{x}|$. Here $\mathbf{x} = (x^1, x^2, x^3)$, so $|\mathbf{x}| = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}$. Then

$$S_{fi} = \frac{iZe^2 m}{V\sqrt{E_i E_f}}\, \bar{u}(p_f, s_f)\gamma^0 u(p_i, s_i) \int \frac{\exp\left[i(p_f - p_i)\cdot x\right] d^4x}{4\pi|\mathbf{x}|}. \tag{16}$$

Let $q = p_f - p_i$. Write $\exp(iq \cdot x) = \exp[i(q^0 x^0 - \mathbf{q} \cdot \mathbf{x})]$. Perform the following integrations:

$$\int \exp(iq^0 x^0) dx^0 = 2\pi\delta(q^0) = 2\pi\delta(E_f - E_i); \tag{17}$$

$$\int \frac{\exp(-i\mathbf{q} \cdot \mathbf{x})}{|\mathbf{x}|} d^3x = \frac{4\pi}{|\mathbf{q}|^2} = \frac{4\pi}{|\mathbf{p_f} - \mathbf{p_i}|^2}; \tag{18}$$

and find

$$S_{fi} = \frac{iZe^2 m}{V\sqrt{E_i E_f}} \frac{\bar{u}(p_f, s_f)\gamma^0 u(p_i, s_i)}{|\mathbf{q}|^2} 2\pi\delta(E_f - E_i). \tag{19}$$

The integral performed in Eq. (18) is the Fourier transform of a generalized function. [7] Use Eq. (19) to calculate the differential cross section.[6] The result is

$$\frac{d\sigma}{d\Omega} = \frac{Z^2\alpha^2(1 - \beta^2 sin^2(\theta/2))}{4\mathbf{p}^2\beta^2 sin^4(\theta/2)} \tag{20}$$

where $\alpha = e^2/4\pi$ is the fine structure constant. This is the Mott cross section. It reduces to the Rutherford formula for small β.

IV. EXTENDED ELECTRON SCATTERING FROM A POINT CHARGE

In the previous section, x was the argument of the wave function and the electron charge spacetime point in an arbitrary Lorentz frame. In

this section, take the electron to be moving in the $+x^3$ direction with a speed β_i. This frame was previously defined to be the m frame. So now x_m is the argument of the wave function, and it is also the center of the electron charge distribution.

Eq. (13) will be modified in two steps to take into account the spatial distribution of the electron charge. First, the interaction takes place at the charge point x'_m, so for a point charge $-Ze > 0$, which is at rest at the origin, the Coulomb potential $A_0(\mathbf{x}) = -Ze/4\pi|\mathbf{x}|$ is replaced by $A_0(\mathbf{x}'_m) = -Ze/4\pi|\mathbf{x}'_m|$. With this modification, the S-matrix for the extended electron, call it S_{FI}, now is

$$S_{FI} = \frac{iZem}{V\sqrt{E_i E_f}}\,\bar{u}(p_f, s_f)\gamma^0 u(p_i, s_i)\int \frac{e\,\exp\left[i(p_f - p_i)\cdot x_m\right]d^4x_m}{4\pi|\mathbf{x}'_m|}. \tag{21}$$

Note that in Eq. (21), p_f and p_i refer to the momenta in the m frame. Introduce $q_m = p_f - p_i$.

Second, replace the electron charge e within the integral by the four-dimensional integral of $de_m = \rho_r(L(\tilde{x}_m))\delta[\gamma_i(\tilde{x}^0_m - \beta_i\,\tilde{x}^3_m]d^4\tilde{x}_m$. So now

$$S_{FI} = \frac{iZem}{V\sqrt{E_i E_f}}\bar{u}(p_f, s_f)\gamma^0 u(p_i, s_i) \int \frac{\exp(+iq_m \cdot x_m) d^4 x_m}{4\pi|\mathbf{x}'_m|} \rho_r(L(\tilde{x}_m))\delta[\gamma_i(\tilde{x}^0_m - \beta_i\, \tilde{x}^3_m] d^4\tilde{x}_m. \quad (22)$$

Change variables from x_m to $x'_m = x_m + \tilde{x}_m$. Then

$$S_{FI} = \frac{iZe^2m}{V\sqrt{E_i E_f}}\bar{u}(p_f, s_f)\gamma^0 u(p_i, s_i) \int \frac{\exp(+iq_m \cdot x'_m) d^4 x'_m}{4\pi|\mathbf{x}'_m|} \int \frac{\exp(-iq_m \cdot \tilde{x}_m)\rho_r(L(\tilde{x}_m))\delta[\gamma_i(\tilde{x}^0_m - \beta_i\, \tilde{x}^3_m]}{e}\, d^4\tilde{x}_m. \quad (23)$$

Notice that $eA_0(\mathbf{x})$, the potential energy of a point electron in a Coulomb field, has been replaced by $\int \rho_r(L(\tilde{x}_m))A_0(\mathbf{x}'_m)d^3\tilde{x}_m$, the potential energy of an extended electron in a Coulomb field. By Eq. (16)

$$S_{FI} = S_{fi}F(q) \quad (24)$$

where S_{fi} is the S-matrix for point electron scattering in a Coulomb potential, and

$$F(q) = \frac{1}{e}\int \exp(-iq_m \cdot \tilde{x}_m)\rho_r(L(\tilde{x}_m))\delta[\gamma_i(\tilde{x}^0_m - \beta_i\, \tilde{x}^3_m]d^4\tilde{x}_m. \quad (25)$$

$F(q)$ is called the electron form factor. The m frame is actually an arbitrary Lorentz frame since β_i can take a range of values from zero up to any number less than one. The direction of motion is also arbitrary, but is conveniently chosen to be the $+x^3$ direction. Thus, the form factor takes the same value in all Lorentz frames. For this reason, the form factor is written $F(q)$ without a subscript on the q. By Eq. (3), or by Eqns. (6), (7), and invariance of the scalar product,

$$F(q) = \frac{1}{e}\int \exp\left(-iq_r \cdot \tilde{x}_r\right)\rho_r(\tilde{x}_r))\delta(x_r^0)d^4\tilde{x}_r = \frac{1}{e}\int \exp\left(+i\mathbf{q}_r \cdot \tilde{\mathbf{x}}_r\right)\rho_r(\tilde{x}_r))d^3\tilde{x}_r. \tag{26}$$

where $q_r = (q_r^0, q_r^1, q_r^2, q_r^3) = L(q_m) = (\gamma_i(q_m^0 - \beta_i q_m^3), q_m^1, q_m^2, \gamma_i(q_m^3 - \beta_i q_m^0))$, and $|\mathbf{q}_r|^2 = (q_r^1)^2 + (q_r^2)^2 + (q_r^3)^2 = (q_m^1)^2 + (q_m^2)^2 + \gamma_i^2(q_m^3 - \beta_i q_m^0)^2$. The form factor is conveniently calculated in the rest frame of the electron; however, $|\mathbf{q}_r|$ is expressed in terms of $|\mathbf{q}_m| = |\mathbf{p}_f - \mathbf{p}_i|$ since it is p_f and p_i which are measured in the lab frame.

V. THE FORM FACTOR

The form factor can be determined from the measured differential cross section in a scattering experiment. Then, the charge density, which is the Fourier transform of $F(q)$, can be calculated. To get a

rough idea of how electron size affects scattering, choose the electron charge density to be given by Eq. (8), so

$$\rho_r(\tilde{x}_r) = \frac{e}{4\pi a^2}\delta(\sqrt{(\tilde{x}_r^1)^2 + (\tilde{x}_r^2)^2 + (\tilde{x}_r^3)^2} - a), \tag{27}$$

and

$$F(q) = \int \exp\left(-iq_r \cdot \tilde{x}_r\right)\delta(\tilde{x}_r^0)\frac{\delta(\sqrt{(\tilde{x}_r^1)^2 + (\tilde{x}_r^2)^2 + (\tilde{x}_r^3)} - a)d^4\tilde{x}_r}{4\pi a^2}. \tag{28}$$

In spherical coordinates, $(\tilde{r}_r)^2 = (\tilde{x}_r^1)^2 + (\tilde{x}_r^2)^2 + (\tilde{x}_r^3)^2$, so that $d^3\tilde{x}_r = (\tilde{r}_r)^2 \sin\tilde{\theta}_r\, d\tilde{\theta}_r\, d\tilde{\phi}_r\, d\tilde{r}_r$. Then

$$F(q) = \int \exp\left(i\tilde{r}_r|\mathbf{q}_r|\cos\tilde{\theta}\right)\frac{\delta(\tilde{r}_r - a)(\tilde{r}_r)^2 \sin\tilde{\theta}_r\, d\tilde{\theta}_r\, d\tilde{\phi}_r\, d\tilde{r}_r}{4\pi a^2}. \tag{29}$$

After integration,

$$F(q) = \frac{\exp\left(i|\mathbf{q}_r|a\right) - \exp\left(-i|\mathbf{q}_r|a\right)}{2i|\mathbf{q}_r|a} = \frac{\sin\left(|\mathbf{q}_r|a\right)}{|\mathbf{q}_r|a} = j_0(|\mathbf{q}_r|a) \tag{30}$$

where j_0 is the spherical Bessel function of order zero.

The differential cross section will be the product of the Mott cross section times $j_0^2(|\mathbf{q}_r|a)$. For $|\mathbf{q}_r|a \ll 1$, $F(q) = sin(|\mathbf{q}_r|a)/|\mathbf{q}_r|a \simeq 1$

and the differential cross section approaches the Mott cross section. Electron size will become evident only at high electron speeds.

It will now be verified that $j_0(|\mathbf{q}_r|a)$ is invariant. Since $q_m = p_f - p_i$, $\mathbf{p}_f = (p_f^1, p_f^2, p_f^3)$, and $\mathbf{p}_i = (0, 0, p_i^3)$,

$$|\mathbf{q}_r|^2 = (p_f^1)^2 + (p_f^2)^2 + \gamma_i^2(p_f^3 - p_i^3 - \beta_i(p_f^0 - p_i^0))^2. \tag{31}$$

Use $p_i^3 = \beta_i p_i^0$, and find $|\mathbf{q}_r|^2 = (p_f^1)^2 + (p_f^2)^2 + \gamma_i^2(p_f^3 - \beta_i p_f^0)^2$. Upon putting over a common denominator,

$$|\mathbf{q}_r|^2 = \frac{(p_f^1)^2 + (p_f^2)^2 + (p_f^3)^2 - 2\beta_i\, p_f^3 p_f^0 + \beta_i^2((p_f^0)^2 - (p_f^1)^2 - (p_f^2)^2)}{1 - \beta_i^2}. \tag{32}$$

Set $\beta_i = p_i^3/p_i^0$, multiply numerator and denominator by $(p_i^0)^2$, and find

$$|\mathbf{q}_r|^2 = \frac{(p_f^0)^2(p_i^0)^2 - 2p_f^3\, p_f^0\, p_i^3\, p_i^0 + (p_f^3)^2(p_i^3)^2 - m^2((p_i^0)^2 - (p_i^3)^2)}{(p_i^0)^2 - (p_i^3)^2}. \tag{33}$$

Use $(p_i^0)^2 - (p_i^3)^2 = m^2$, and finally find

$$|\mathbf{q}_r|^2 = \frac{(p_f \cdot p_i)^2 - m^4}{m^2}. \tag{34}$$

Since $|\mathbf{q}_r|$ contains a scalar product and the rest mass, $|\mathbf{q}_r|$ and $j_0(|\mathbf{q}_r|a)$ are invariant.

To facilitate the calculation of the form factor from experimental data, $|\mathbf{q}_r|$ can be written in terms of the initial electron momentum magnitude $|\mathbf{p_i}|$, the angle of scattering θ, and the initial speed of the electron β_i. From the delta function in Eq. (17), $E_f = E_i$, or $p_f^0 = p_i^0$. Thus, Eq. (31) can be written

$$|\mathbf{q}_r|^2 = (p_f^1)^2 + (p_f^2)^2 + \gamma_i^2(p_f^3 - p_i^3)^2. \tag{35}$$

Also, it follows from $E_f = E_i$ that the final speed of the electron $\beta_f = \beta_i$, and $|\mathbf{p}_f| = |\mathbf{p}_i|$. Use $(p_f^1)^2 + (p_f^2)^2 = |\mathbf{p}_f|^2 \sin\theta = |\mathbf{p}_i|^2 \sin\theta$ and $p_f^3 = |\mathbf{p}_f| \cos\theta = |\mathbf{p}_i| \cos\theta$. Then

$$(1 - \beta_i^2)|\mathbf{q}_r|^2 = |\mathbf{p_i}|^2 \sin^2\theta\,(1 - \beta_i^2) + |\mathbf{p_i}|^2(\cos\theta - 1)^2, \tag{36}$$

and

$$(1 - \beta_i^2)|\mathbf{q}_r|^2 = 2|\mathbf{p_i}|^2(1 - \cos\theta) - \beta_i^2|\mathbf{p_i}|^2 \sin^2\theta. \tag{37}$$

Use $1 - \cos\theta = 2\sin^2(\theta/2)$ and $\sin\theta = 2\sin(\theta/2)\cos(\theta/2)$, and find

$$|\mathbf{q}_r|^2 = \frac{4|\mathbf{p_i}|^2 \sin^2(\theta/2)(1 - \beta_i^2\cos^2(\theta/2))}{(1 - \beta_i^2)}. \tag{38}$$

VI. ALTERNATIVE ELECTRON CHARGE DENSITY

As another example, let the electron charge distribution be

$$\rho_r(\tilde{x}_r) = \frac{3e}{4\pi a^3} H(a - \sqrt{(\tilde{x}_r^1)^2 + (\tilde{x}_r^2)^2 + (\tilde{x}_r^3)^2}\,) = \frac{3e}{4\pi a^3} H(a - \tilde{r}_r) \quad (39)$$

where $H(a - \tilde{r}_r)$, the unit step function, is defined by $H(a - \tilde{r}_r) = 1$ for $a > \tilde{r}_r$ and $H(a - \tilde{r}_r) = 0$ for $a < \tilde{r}_r$. This corresponds to a uniform spherical charge density of radius a. For the new electron charge distribution, it can again be shown that $\int \rho_r(\tilde{x}_r)\delta(\tilde{x}_r^0) d^4\tilde{x}_r$ equals e.

For the above electron moving in the $+x^3$ direction,

$$\rho_r(L(\tilde{x}_m)) = \frac{3e}{4\pi a^3}\, H(a - \sqrt{(\tilde{x}_m^1)^2 + (\tilde{x}_m^2)^2 + \gamma_i^2(\tilde{x}_m^3 - \beta_i \tilde{x}_m^0)^2}). \quad (40)$$

It can be shown that the four-dimensional integral of $\rho_r(L(\tilde{x}_m))$ times $\delta[\gamma_i(\tilde{x}_m^0 - \beta_i \tilde{x}_m^3)]$ is again e. For an electron, which moves in the x^3 direction, and is scattered by the Coulomb potential of a point charge at the origin, the S-matrix is again the product of the S-matrix of the point electron theory times the electron form factor. For this electron charge distribution, the form factor is given by

$$F(q) = \frac{3}{(|\mathbf{q}_r|a)^3}(\sin(|\mathbf{q}_r|a) - |\mathbf{q}_r|a\cos(|\mathbf{q}_r|a)) = \frac{3}{|\mathbf{q}_r|a}j_1(|\mathbf{q}_r|a) \quad (41)$$

with $j_1(|\mathbf{q}_r|a)$ being the spherical Bessel function of order one.

VII. DISCUSSION

The purpose of the paper is to investigate how electron size and structure affect electron scattering. Electron structure enters the theory when the differential of the electron charge is replaced by a function multiplied by a four-dimensional differential volume element. The result is that the S-matrix is the product of the S-matrix of a point electron times the electron form factor. The form factor becomes noticeable only at high electron speeds.

The classical picture of the electron as a charge distribution in space can be replaced by a quantum mechanical interpretation in terms of probability amplitudes. The S-matrix is a probability amplitude. As such, it is the sum of products of probability amplitudes. In QED, the S-matrix contains the probability amplitude for the point electron to interact with the potential at the charge point. This probability amplitude is replaced by a sum over probability amplitudes of the extended

electron to interact with the potential over a distribution of charge points.

The two electron charge densities used in this paper were chosen in order to get a rough idea of the dependence of the form factor on the electron structure and on electron radius. The actual electron charge density, which is the Fourier transform of $eF(q)$, can be calculated from the experimentally determined electron form factor.

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