SOME INTEGRALS AND A LITTLE E&M (A REVIEW)

This paper presents a list of integrals that will be useful in the papers that follow.

I. BOX NORMALIZATION

Box normalization is introduced and briefly explained in the review paper on the Schrödinger equation. The integral to be evaluated is

$$I = \int_{-L/2}^{+L/2} \phi^*(x,t)\phi(x,t)\,dx \tag{1}$$

where $\phi(x,t) = \exp[+i(kx - \omega t)]/L$. The wave vector $k = 2\pi n/L$, where n is an integer, L is the length of the one-dimensional box, and $\omega = \hbar k^2/(2m)$. For $n_2 \neq n_1$,

$$I = \int_{-L/2}^{+L/2} \frac{\exp[+i2\pi x(n_2 - n_1)/L - i(\omega_2 - \omega_1)t]}{L}\,dx \propto \frac{\sin[\pi(n_2 - n_1)]}{\pi(n_2 - n_1)} = 0. \tag{2}$$

When $n_2 = n_1$, then $\omega_2 = \omega_1$, and $I = (\int \exp(0)\,dx)/L = 1$. Thus, $I = \delta_{k_2,k_1}$ where δ_{k_2,k_1}, the Kroneckor delta, is defined to be zero if $k_2 \neq k_1$ and defined to equal 1 if $k_2 = k_1$.

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II. THE DELTA FUNCTION

The delta function $\delta(x)$ is defined by

$$\int_{-\infty}^{\infty} g(x)\delta(x)\,dx = g(0) \tag{3}$$

where $g(x)$ is a function that is continuous at $x = 0$. In the integral below, introduce $y = x - a$

$$\int_{-\infty}^{\infty} g(x)\delta(x-a)\,dx = \int_{-\infty}^{\infty} g(y+a)\delta(y)\,dy = g(a) \tag{4}$$

where $g(x)$ is continuous at $x = a$. In the integral below, introduce $y = -x$

$$\int_{-\infty}^{\infty} g(x)\delta(-x)\,dx = \int_{+\infty}^{-\infty} g(-y)\delta(y)\,(-dy) = g(0). \tag{5}$$

We conclude that $\delta(x) = \delta(-x)$, i.e., the delta function is an even function. Next, observe that

$$\int_{-\infty}^{\infty} g(x)x\delta(x)\,dx = g(x)x|_{x=0} = 0, \tag{6}$$

so we conclude $x\delta(x) = 0$.

The definition of the Fourier transform of the function $g(x)$ is

$$G(\alpha) = \int_{-\infty}^{\infty} g(x)exp(-i\alpha x)\,dx\,, \tag{7}$$

and the inverse Fourier transform is then

$$g(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\alpha) exp(+i\alpha x)\, d\alpha\,. \tag{8}$$

So the Fourier transform of $\delta(x)$ is given by

$$G(\alpha) = \int_{-\infty}^{\infty} \delta(x) exp(-i\alpha x)\, dx \; = \exp(0) = 1. \tag{9}$$

Therefore, the the inverse Fourier transform of 1 is

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 1\, exp(+i\alpha x)\, d\alpha\,, \tag{10}$$

and since $\delta(x)$ is an even function,

$$\delta(x) = \delta(-x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} exp(-i\alpha x)\, d\alpha\,. \tag{11}$$

The Fourier transform of $\delta(x-a)$ is given by

$$G(\alpha) = \int_{-\infty}^{\infty} \delta(x-a) exp(-i\alpha x)\, dx \; = \exp(-i\alpha a), \tag{12}$$

So the inverse Fourier transform of $\exp(-i\alpha a)$ is

$$\delta(x-a) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(-i\alpha a)\, exp(+i\alpha x)\, d\alpha\,. \tag{13}$$

These formulas will be used often in the papers that follow.

III. GENERALIZED FUNCTIONS AND MORE FOURIER TRANSFORMS

The reader is encouraged to consult a book on generalized functions. I have left out too much, and besides, generalized functions are fun. Possibly the best book to start with is *Introduction to Fourier Analysis and Generalized Functions* by M. J. Lighthill(less than 100 pages). Unfortunately, the 2π is in the exponent in this book. The book I like is *Generalized Functions* by D. S. Jones(over 400 pages).

Introduce a function $\gamma(x)$ which is infinitely differentiable, and $\gamma(x) \to 0$ as $x \to \pm\infty$. A generalized function $g(x)$ is defined so that the integral

$$\int_{-\infty}^{\infty} g(x)\gamma(x)\, dx \tag{14}$$

exists and is finite. The delta function is an example of a generalized function. Some ordinary functions can be defined as generalized functions. The definition requires that the discontinuities of the ordinary function be smoothed out so that all derivatives exist. In addition, the generalized function $\to 0$ as $x \to \pm\infty$. As an example, consider the function $\text{sgn}(x) = -1$ for $x < 0$, $\text{sgn}(0) = 0$, and $\text{sgn}(x) = +1$ for $x > 0$. As a generalized function all derivatives exist. Put the derivative of $\text{sgn}(x)$ in Eq. (14), and using integration by parts, find

$$\int_{-\infty}^{\infty} \frac{d\,\mathrm{sgn}(x)}{dx}\gamma(x)\,dx = \mathrm{sgn}(x)\gamma(x)|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \mathrm{sgn}(x)(\gamma(x))'dx =$$
$$- - \int_{-\infty}^{0} (\gamma(x))'dx - \int_{0}^{\infty} (\gamma(x))'dx = \gamma(x)|_{-\infty}^{0} - \gamma(x)|_{0}^{\infty} = 2\gamma(0)\,. \tag{15}$$

By Eq. (3), we see $(\mathrm{sgn}(x))' = 2\delta(x)$.

Recall Eq. (7) where $G(\alpha)$ is defined to be the Fourier transform of $g(x)$. We now find the Fourier transform of the derivative of $g(x)$. Using integration by parts,

$$\int_{-\infty}^{\infty} \frac{dg(x)}{dx} exp(-i\alpha x)\,dx = g(x)\exp(-i\alpha x)|_{-\infty}^{+\infty} - \int_{-\infty}^{\infty} g(x)\frac{d\,[exp(-i\alpha x)]}{dx}\,dx =$$
$$i\alpha \int_{-\infty}^{\infty} g(x)\exp(-i\alpha x)dx = i\alpha G(\alpha). \tag{16}$$

So the Fourier transform of $g'(x)$ is $i\alpha G(\alpha)$ if $G(\alpha)$ is the Fourier transform of $g(x)$. This result will be used to find the Fourier transform of the function $\mathrm{sgn}(x)$.

$$\int_{-\infty}^{+\infty} (\mathrm{sgn}(x))' \exp(-i\alpha x)dx = \int_{-\infty}^{+\infty} 2\delta(x)\exp(-i\alpha x)dx = 2 = i\alpha G(\alpha). \tag{17}$$

So the Fourier transform of of $\mathrm{sgn}(x)$ is $G(\alpha) = 2/(i\alpha)$.

The above result is used in the evaluation of Eq. (14) in the paper on Mott Rutherford scattering. The integral is to be evaluated over all space. The integral in Cartesian coordinates is best evaluated in spherical coordinates as below

$$\int \frac{\exp{(i\mathbf{q}\cdot\mathbf{x})}}{|\mathbf{x}|}\,d^3x = \int_0^{+\infty}\int_0^{2\pi}\int_0^{\pi} \frac{\exp{(i|\mathbf{q}|r\cos\theta)}}{r} r^2 \sin\theta d\theta\, d\phi\, dr = \\ 2\pi \int_0^{+\infty} \frac{\exp{(i|\mathbf{q}|r)} - \exp{(-i|\mathbf{q}|r)}}{i|\mathbf{q}|r^2} r^2\, dr\,, \quad (18)$$

where the spherical coordinate $r = |\mathbf{x}|$. Replace r by $-r$ in the first exponential. Then

$$\int_0^{-\infty} \exp{(-i|\mathbf{q}|r)}(-)dr = \int_{-\infty}^{0} \exp{(-i|\mathbf{q}|r)}dr\,. \quad (19)$$

So now

$$\int \frac{\exp{(i\mathbf{q}\cdot\mathbf{x})}}{|\mathbf{x}|}\,d^3x = \frac{2\pi}{i|\mathbf{q}|}\int_{-\infty}^{0} \exp{(-i|\mathbf{q}|r)}dr - \frac{2\pi}{i|\mathbf{q}|}\int_0^{\infty} \exp{(-i|\mathbf{q}|r)}dr \\ = \frac{-2\pi}{i|\mathbf{q}|}\int_{-\infty}^{\infty} \mathrm{sgn}(r)\exp{(-i|\mathbf{q}|r)}dr. \quad (20)$$

Recall that the Fourier transform of $\mathrm{sgn}(r)$ is $2/i|\mathbf{q}|$, so finally

$$\int \frac{\exp{(i\mathbf{q}\cdot\mathbf{x})}}{|\mathbf{x}|}\,d^3x = \frac{-4\pi}{(i|\mathbf{q}|)^2} = \frac{4\pi}{|\mathbf{q}|^2}. \quad (21)$$

We will now show

$$\int_{-\infty}^{\infty} \delta(\alpha)\delta(x-\alpha)\,d\alpha = \delta(x)\,. \quad (22)$$

To do this, we will use the theorem

$$\int_{-\infty}^{\infty} \Psi(\alpha) G(x-\alpha)\, d\alpha = 2\pi \int_{-\infty}^{\infty} \psi(t) g(t) \exp(-itx)\, dt, \tag{23}$$

where $\Psi(\alpha)$ is the Fourier transform of $\psi(t)$, and $G(\alpha)$ is the Fourier transform of $g(t)$. First, let $\psi(t) = 1$, and let $g(t) = 1$. Then $\Psi(\alpha) = 2\pi\delta(\alpha)$, and $G(\alpha) = 2\pi\delta(\alpha)$. Substitute this in Eq. (23),

$$(2\pi)^2 \int_{-\infty}^{\infty} \delta(\alpha)\delta(x-\alpha)\, d\alpha = 2\pi \int_{-\infty}^{\infty} \exp(-itx)\, dt = (2\pi)^2 \delta(x)\,. \tag{24}$$

Cancel $(2\pi)^2$, and find Eq. (22).

Next, we will show

$$\int_{-\infty}^{\infty} \delta(\alpha - y)\delta(x-\alpha)\, d\alpha = \delta(x-y)\,. \tag{25}$$

Let $\psi(t) = \exp(iyt)$, and let $g(t) = 1$. Then, $G(\alpha) = 2\pi\delta(\alpha)$, and

$$\Psi(\alpha) = \int_{-\infty}^{\infty} \exp(iyt) \exp(-i\alpha t)\, dt = 2\pi\delta(\alpha - y). \tag{26}$$

By Eq. (23),

$$(2\pi)^2 \int_{-\infty}^{\infty} \delta(\alpha - y)\delta(x-\alpha))\, d\alpha =$$
$$2\pi \int_{-\infty}^{\infty} \exp(iyt) \exp(-itx)\, dt = (2\pi)^2 \delta(x-y)\,. \tag{27}$$

Thus, we recover Eq. (25).

We will next show

$$\int_{-\infty}^{+\infty} \Psi(\alpha)\delta(x-\alpha)\delta(\alpha+y)\,d\alpha = \Psi(x)\delta(x+y). \tag{28}$$

Write the second delta function as an integral of the exponential, thus

$$\begin{aligned}
&\int_{-\infty}^{+\infty} \Psi(\alpha)\delta(x-\alpha)\delta(\alpha+y)\,d\alpha = \\
&\qquad \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \Psi(\alpha)\delta(x-\alpha)\exp\left(i(\alpha+y)t\right)\frac{dt}{2\pi}\,d\alpha = \\
&\qquad\qquad \int_{-\infty}^{+\infty} \Psi(x)\exp\left(i(x+y)t\right)\frac{dt}{2\pi} = \Psi(x)\delta(x+y).
\end{aligned} \tag{29}$$

This equation is used in the paper on Møller scattering. The equation to be integrated there is

$$\int \frac{1}{q^2+i\epsilon}\,\delta^4(\Delta p - q)\delta^4(\Delta P + q)\,d^4q. \tag{30}$$

There are four similar integrals to be performed. We do one of them.

$$\begin{aligned}
&\int \frac{1}{(q^0)^2-(q^1)^2-(q^2)^2-(q^3)^2+i\epsilon}\,\delta(\Delta p^1 - q^1)\delta(\Delta P^1 + q^1)\,dq^1 = \\
&\qquad \frac{1}{(q^0)^2-(\Delta p^1)^2-(q^2)^2-(q^3)^2+i\epsilon}\,\delta(\Delta p^1 + \Delta P^1).
\end{aligned} \tag{31}$$

After doing the other three integrations, Eq. (30) equals

$$\frac{1}{(\Delta p)^2 + i\epsilon}\,\delta^4(\Delta p + \Delta P). \tag{32}$$

IV. CONTOUR INTEGRATION

The integrals in this section are repeatedly used in the papers that follow. They are first used in the paper on the photon propagator, and then in the paper on the Feynman propagator.

In this section, we evaluate integrals on the real axis which have infinite limits of integration. The approach taken is to define a closed contour which includes our integral. The contour is chosen so that the integral on the remainder of the contour is zero or can be determined. We then use residues to determine our integral.

We wish to evaluate the integral below

$$I = \int_{-\infty}^{+\infty} \frac{f(x)\exp(-imx)}{x - (x_0 - i\epsilon)}\,dx \tag{33}$$

where $f(x)$ is infinitely differentiable, m is positive, x_0 is a real constant, and ϵ is a small positive constant. Notice that there is a simple pole below the x axis. Let S_M be a semi-circle below the x axis of radius M with the radius originating at the origin. Let the contour run along the real x axis from $-M$ to $+M$, and then from $+M$ to $-M$ along S_M.

The integral I_C, which is defined below, is evaluated using the residue theorem

$$I_C = \int_{-M}^{+M} \frac{f(x)\exp(-imx)}{x-(x_0-i\epsilon)}\,dx + \int_{S_M} \frac{f(z)\exp(-imz)}{z-(x_0-i\epsilon)}\,dz = \\ -2\pi i \mathrm{Res}(z = x_0 - i\epsilon) \quad (34)$$

where the residue at $z = x_0 - i\epsilon$ is symbolized by $\mathrm{Res}(x_0 - i\epsilon)$ and is equal to $f(x_0 - i\epsilon)\exp[-im(x_0 - i\epsilon)]$. The - sign on the right hand side of Eq. (34) occurs because the contour is traversed in a clockwise sense. Notice that the path of integration has been chosen so that the integral on S_M vanishes as $M \to \infty$. Thus,

$$I = \lim_{M\to\infty} I_C = -2\pi i Res(x_0 - i\epsilon) = -2\pi i f(x_0 - i\epsilon)\exp[-im(x_0 - i\epsilon)]. \quad (35)$$

When m is negative, we pick S_M to be the semi-circle above the x axis. Then the integral on S_M vanishes as $(M) \to +\infty$. Thus,

$$I = \lim_{M\to\infty} I_C = +2\pi i \mathrm{Res}(z) = 0 \quad (36)$$

since there are no poles within the contour of integration. The + sign occurs because the contour is traversed in a counter-clockwise sense.

Next, consider the integral

$$I = \int_{-\infty}^{+\infty} \frac{f(x)\exp(-imx)}{x + (x_0 - i\epsilon)}\, dx \tag{37}$$

where m is negative. The simple pole is above the x axis. Evaluate the integral

$$I_C = \int_{-M}^{+M} \frac{f(x)\exp(-imx)}{x + (x_0 - i\epsilon)}\, dx + \int_{S_M} \frac{f(z)\exp(-imz)}{z + (x_0 - i\epsilon)}\, dz = 2\pi i \mathrm{Res}(z = -(x_0 - i\epsilon)) \tag{38}$$

Again, the integral on S_M vanishes, and

$$I = \lim_{M\to\infty} I_C = +2\pi i f(-x_0 + i\epsilon)\exp[-im(-x_0 + i\epsilon)]. \tag{39}$$

When m is positive, take S_M below the x axis, and find

$$I = \lim_{M\to\infty} I_C = -2\pi i \mathrm{Res}(z) = 0. \tag{40}$$

Next, consider the integral

$$I = \int_{-\infty}^{+\infty} \frac{f(x)\exp(-imx)}{x - x_0}\, dx \tag{41}$$

where the pole is now on the x axis. Let I_C be the integral of $f(z)\exp(-imz)/(z - x_0)$ where for m positive, the path of integration is from $-M$ to $x_0 - \epsilon$, from $x_0 - \epsilon$ to $x_0 + \epsilon$ along a semi-circle S_ϵ of radius ϵ, which is below the x axis, and then from $x_0 + \epsilon$ to $+M$, and from M to $-M$ along a semi-circle S_M below the x axis of radius M. As before, when $M \to \infty$, the integral along S_M vanishes. Define the Cauchy principle value by

$$P \int_{-\infty}^{+\infty} \frac{f(x) \exp(-imx)}{x - x_0} \, dx = \lim_{\epsilon \to 0} \Big[\int_{-\infty}^{x_0 - \epsilon} \frac{f(x) \exp(-imx)}{x - x_0} \, dx + \int_{x_0 + \epsilon}^{+\infty} \frac{f(x) \exp(-imx)}{x - x_0} \, dx \Big] \tag{42}$$

Then in the limit as $M \to \infty$, and $\epsilon \to 0$

$$I_C \to P \int_{-\infty}^{+\infty} \frac{f(x) \exp(-imx)}{x - x_0} \, dx + \int_{S_\epsilon} \frac{f(x) \exp(-imx)}{x - x_0} \, dx = 0 \tag{43}$$

since there are no poles inside the contour of integration. The second integral is $+\pi i \mathrm{Res}(x = x_0)$. So finally,

$$P \int_{-\infty}^{+\infty} \frac{f(x) \exp(-imx)}{x - x_0} \, dx = -\pi i f(+x_0) \exp(-imx_0) \tag{44}$$

When m is negative, the path of integration is similar, but S_M and S_ϵ are above the x axis. The result is

$$P \int_{-\infty}^{+\infty} \frac{f(x) \exp(-imx)}{x - x_0} \, dx = +\pi i f(+x_0) \exp(-imx_0). \tag{45}$$

V. CHANGE OF VARIABLE

A spacetime point in the rest frame is represented by $x_r^\mu = (x_r^0, x_r^1, x_r^2, x_r^3)$ where $x_r^0 = ct_r$, c is the speed of light, and t_r is the time in the rest frame. In a frame of reference which moves with a speed v relative to the rest frame in the $+x_r^3$ direction, a spacetime point is represented by $x_m^\mu = (x_m^0, x_m^1, x_m^2, x_m^3)$. The two sets of coordinates are related by the following Lorentz transformation: $x_r^1 = x_m^1$, $x_r^2 = x_m^2$, $x_r^0 = \gamma(x_m^0 - \beta x_m^3)$, and $x_r^3 = \gamma(x_m^3 - \beta x_m^0)$ where $\beta = v/c$ and $\gamma = 1/\sqrt{1-\beta^2}$. Since $x_r^1 = x_m^1$, and $x_r^2 = x_m^2$, we shall concentrate on integrals in the x^0, x^3 plane.

Green's theorem states

$$\int_\Gamma \big(P(x,y)dx + Q(x,y)dy\big) = \int_R \Big(\frac{\partial Q(x,y)}{\partial x} - \frac{\partial P(x,y)}{\partial y}\Big) dx\,dy \qquad (46)$$

where the second integral is over a two dimensional region R in the x, y plane, and the first integral is along a curve Γ, which encloses the region R. If $Q = x$, and $P = 0$, then Green's theorem gives

$$\int_\Gamma x dy = \int_R dx dy = A \qquad (47)$$

where A is the area of the region R, which is bounded by the curve Γ. Apply this result to the x_r^0, x_r^3 plane and find

$$A = \int_{R_r} dx_r^0\, dx_r^3 = \int_{\Gamma_r} x_r^0\, dx_r^3\,. \tag{48}$$

Lorentz transform to the moving frame and get

$$A = \int_{\Gamma_m} \gamma(x_m^0 - \beta x_m^3)\gamma(dx_m^3 - \beta dx_m^0) = \int_{\Gamma_m} \gamma^2[(x_m^0 - \beta\, x_m^3)dx_m^3 - (\beta\, x_m^0 - \beta^2\, x_m^3)dx_m^0]. \tag{49}$$

Use Green's theorem to make the identification $P = -\gamma^2(\beta\, x_m^0 - \beta^2\, x_m^3)$ and $Q = \gamma^2(x_m^0 - \beta\, x_m^3)$. Then

$$A = \int_{R_m} \gamma^2(1-\beta^2)dx_m^0\, dx_m^3 = \int_{R_m} dx_m^0\, dx_m^3 = \int_{R_r} dx_r^0\, dx_r^3\,. \tag{50}$$

Since $dx_r^1\, dx_r^2 = dx_m^1\, dx_m^2$, we conclude that

$$\int dx_r^0 dx_r^1 dx_r^2 dx_r^3 = \int dx_m^0 dx_m^1 dx_m^2 dx_m^3\,. \tag{51}$$

We will apply the above result to integrals over all of spacetime.

The integral that will be needed in the non-review papers takes the form

$$\int f(x_r^0, x_r^1, x_r^2, x_r^3) dx_r^0\, dx_r^1\, dx_r^2\, dx_r^3 = \int f[\gamma(x_m^0 - \beta x_m^3), x_m^1, x_m^2, \gamma(x_m^3 - \beta x_m^0)] dx_m^0\, dx_m^1\, dx_m^2\, dx_m^3. \tag{52}$$

A rigorous proof of Eq. (52) in a more general situation can be found in advanced calculus books.

VI. A LITTLE E&M

We will treat the electron as an extended charge by replacing the electron charge e by a Lorentz invariant four dimensionl integral of the form of Eq. (52). For a charge at rest, the volume integral of the charge density yields the total charge. As an example, consider a specific charge distribution , namely, a line charge at rest, which has a proper length L and which lies along the x^3 axis. Denote the charge points by $x_r^{\mu'} = (x_r^{0\,\prime}, x_r^{1\,\prime} = 0, x_r^{2\,\prime} = 0, x_r^{3\,\prime})$.The left end is at $x_r^{3\,\prime} = 0$ and the right end is at $x_r^{3\,\prime} = L$. For the charge at rest, the charge density is $\rho_r(x_r') = (e/L)\delta(x_r^{1\,\prime})\delta(x_r^{2\,\prime})[H(x_r^{3\,\prime}) - H(x_r^{3\,\prime} - L)]$ where H is the unit step function, defined by $H(x) = 1$ for $x > 0$ and $H(x) = 0$ for $x < 0$. Perform the volume integral over the charge density and find

$$\int (e/L)\delta(x_r^{1'})\,\delta(x_r^{2'})[H(x_r^{3'}) - H(x_r^{3'} - L)]d^3x_r' = (e/L)\int_0^L dx_r^{3'} = e. \tag{53}$$

Let $x_r^\mu = (x_r^0, x_r^1 = 0, x_r^2 = 0, x_r^3)$ denote the spacetime point of the center of the charge distribution. Introduce $\tilde{x} = x' - x$. Change variables in Eq. (53) from x' to $\tilde{x}$

$$\int (e/L)\delta(\tilde{x}_r^1)\,\delta(\tilde{x}_r^2)[H(\tilde{x}_r^3 + L/2) - H(\tilde{x}_r^3 - L/2)]d^3\tilde{x}_r =$$

$$\int \rho_r(\tilde{x}_r)d\tilde{x}_r^3 = (e/L)\int_{-L/2}^{+L/2} d\tilde{x}_r^3 = e. \quad (54)$$

To put Eq. (54) in the form of Eq. (52), note that $\int \delta(\tilde{x}^0)d\tilde{x}^0 = 1$, so $\int \rho_r(\tilde{x}_r)\delta(\tilde{x}_r^0)\,d^4\tilde{x}_r = e$. We tentatively identify $f(\tilde{x}_r)$ with $\rho_r(\tilde{x}_r)\delta(\tilde{x}_r^0)$. The meaning attached to $\tilde{x}_r^0 = 0$ is that the clock at the charge point x'_r reads the same as the clock at x_r in the rest frame. The clocks in the rest frame are synchronized.

We now go on to show that in the moving frame, our function f satisfies the right hand side of Eq. (52).

$$\int \rho_r(\tilde{x}_m)\delta[\gamma(\tilde{x}_m^0 - \beta\,\tilde{x}_m^3]d^4\tilde{x}_m = \int (e/L)\delta(\tilde{x}_m^1)\,\delta(\tilde{x}_m^2)$$

$$\big(H[\gamma(\tilde{x}_m^3 - \beta\tilde{x}_m^0) + L/2] - H[\gamma(\tilde{x}_m^3 - \beta\tilde{x}_m^0) - L/2]\big)\delta[\gamma(\tilde{x}_m^0 - \beta\tilde{x}_m^3)]d^4\tilde{x}_m =$$

$$(e/L)\int [H(\tilde{x}_m^3/\gamma + L/2) - H(\tilde{x}_m^3/\gamma - L/2)]d\tilde{x}_m^3/\gamma = e \quad (55)$$

In the non-review papers, an element of the electron charge in the m frame, de_m, will be replaced by $\rho(\tilde{x}_m)\delta[\gamma(\tilde{x}_m^0 - \beta\tilde{x}_m^3)]d^4\tilde{x}_m$.

VII. COMMENT

The presentation has been too brief. The reader is encouraged to consult a book on contour integration in the complex plane. I have used *Mathematical Methods for Physicists* by George Arfken and *Functions of Complex Variables* by Philip Franklin. I have also consulted *Advanced Calculus* by David Widder.