PHOTON PROPAGATOR (A REVIEW)

I. THE VECTOR POTENTIAL

The inhomogeneous wave equation for the four-dimensional vector potential $A^\mu = (A^0 = \Phi, A^1, A^2, A^3)$ is

$$\frac{\partial^2 A^\mu(x)}{\partial (x^0)^2} - \nabla^2 A^\mu(x) = J^\mu(x) \tag{1}$$

where Φ is the scalar potential, $J^\mu = (c\rho, J^1, J^2, J^3)$ is the four-dimensional current density, and ρ is the charge density. Eq. (1) can be derived from Maxwell's equations and by invoking the Lorenz condition $\partial_\mu A^\mu = 0$, i.e.,

$$\frac{\partial A^0}{\partial x^0} + \frac{\partial A^1}{\partial x^1} + \frac{\partial A^2}{\partial x^2} + \frac{\partial A^3}{\partial x^3} = 0. \tag{2}$$

Introduce the function $D_F(x-y)$, which is defined as the solution to

$$\frac{\partial^2 D_F(x-y)}{\partial (x^0)^2} - \nabla^2 D_F(x-y) = \delta^4(x-y). \tag{3}$$

It will be verified that the solution to Eq. (1) is

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$$A^\mu(x) = \int D_F(x-y) J^\mu(y) d^4\, y \tag{4}$$

where the integral is over all spacetime. Substitute Eq. (4) into Eq. (1), and find

$$\frac{\partial^2 A^\mu(x)}{\partial(x^0)^2} - \nabla^2 A^\mu(x) = \int \left(\frac{\partial^2 D_F(x-y)}{\partial(x^0)^2} - \nabla^2 D_F(x-y)\right) J^\mu(y) d^4\, y = \int \delta^4(x-y) J^\mu(y)\, d^4\, y = J^\mu(x)\,. \tag{5}$$

Thus, we recover Eq. (1).

II. THE PHOTON PROPAGATOR

We shall write $D_F(x-y)$ as the Fourier transform of $D_F(q)$, i.e.,

$$D_F(x-y) = \int \exp\left(-iq\cdot(x-y)\right) D_F(q) \frac{d^4 q}{(2\pi)^4} \tag{6}$$

where $q = (q^0, q^1, q^2, q^3)$ is the four-dimensional wave vector. We shall now determine $D_F(q)$. Substitute Eq. (6) in Eq. (3), and find

$$\frac{\partial^2 D_F(x-y)}{\partial(x^0)^2} - \nabla^2 D_F(x-y) = \int \left(\frac{\partial^2}{\partial(x^0)^2} - \nabla^2\right) \exp\left(-iq\cdot(x-y)\right) D_F(q) \frac{d^4\, q}{(2\pi)^4} = \int \left(-(q^0)^2 + |\mathbf{q}|^2\right) \exp\left(-iq\cdot(x-y)\right) D_F(q) \frac{d^4\, q}{(2\pi)^4}. \tag{7}$$

Use

$$\int \exp\left(-iq\cdot(x-y)\right)\frac{d^4\,q}{(2\pi)^4} = \delta^4(x-y) \tag{8}$$

to show

$$\int \Big[-\big((q^0)^2 - |\mathbf{q}|^2\big)D_F(q) - 1\Big]\exp\left(-iq\cdot(x-y\right)\frac{d^4\,q}{(2\pi)^4} = 0, \tag{9}$$

so

$$((q^0)^2 - |\mathbf{q}|^2)D_F(q) = -1. \tag{10}$$

A solution to Eq. (10) is $D_F(q) = -1/((q^0)^2 - |\mathbf{q}|^2)$, but this is not the most general solution. The most general solution is

$$D_F(q) = \frac{-1}{(q^0)^2 - |\mathbf{q}|^2} + C_1\delta(q^0 - |\mathbf{q}|) + C_2\delta(q^0 + |\mathbf{q})). \tag{11}$$

This can be verified by multiplying Eq. (11) by $(q^0)^2 - |\mathbf{q}^2|$ and using $(q^0 - |\mathbf{q}|)\delta(q^0 - |\mathbf{q}|) = 0$ and $(q^0 + |\mathbf{q}|)\delta(q^0 + |\mathbf{q}|) = 0$; Eq. (10) is recovered. Use partial fractions to show

$$\frac{-1}{(q^0)^2 - |\mathbf{q}|^2} = \frac{-1}{2|\mathbf{q}|}\Big(\frac{1}{q^0 - |\mathbf{q}|} - \frac{1}{q^0 + |\mathbf{q}|}\Big), \tag{12}$$

so

$$D_F(q) = \frac{-1}{2|\mathbf{q}|}\Big(\frac{1}{q^0 - |\mathbf{q}|}\Big) + C_1\delta(q^0 - |\mathbf{q}|) + \frac{1}{2|\mathbf{q}|}\Big(\frac{1}{q^0 + |\mathbf{q}|}\Big) + C_2\delta(q^0 + |\mathbf{q}|). \tag{13}$$

The integral

$$I = \int_{-\infty}^{+\infty} D_F(q)\exp\left(-iq^0(x^0 - y^0)\right)dq^0 = I_1 + I_2 \tag{14}$$

will be evaluated using reasonable physical constraints, which in turn, will determine the constants C_1 and C_2. Here

$$I_1 = \int_{-\infty}^{+\infty} \frac{+1}{2|\mathbf{q}|}\Big(\frac{-1}{q^0 - |\mathbf{q}|} + \frac{1}{q^0 + |\mathbf{q}|}\Big)\exp\left(-iq^0(x^0 - y^0)\right)dq^0, \tag{15}$$

and

$$I_2 = \int_{-\infty}^{+\infty} [C_1\delta(q^0 - |\mathbf{q}|) + C_2\delta(q^0 - |\mathbf{q}|)]\exp\left(-iq^0(x^0 - y^0)\right)dq^0 =$$
$$C_1\exp\left(-i|\mathbf{q}|(x^0 - y^0)\right) + C_2\exp\left(+i|\mathbf{q}|(x^0 - y^0)\right). \tag{16}$$

To evaluate I_1 using contour integration, introduce the closed contour where ϵ is a small positive constant:

(1) from $-M$ to $-|\mathbf{q}| - \epsilon$ along the q^0 axis

(2) from $-|\mathbf{q}| - \epsilon$ to $-|\mathbf{q}| + \epsilon$ along a semi-circle of radius ϵ, which is centered at $-|\mathbf{q}|$, and is below the q^0 axis.

(3) from $-|\mathbf{q}| + \epsilon$ to $+|\mathbf{q}| - \epsilon$ along the q^0 axis

(4) from $+|\mathbf{q}| - \epsilon$ to $+|\mathbf{q}| + \epsilon$ along a semi-circle of radius ϵ, which is centered at $+|\mathbf{q}|$, and is below the q^0 axis.

(5) from $+|\mathbf{q}| + \epsilon$ to $+M$ along the q^0 axis

(6) $+M$ to $-M$ along a semi-circle S_M of radius M, which is centered at the origin of the q^0 axis and is below the axis.

This contour will be referred to as contour C1. There are no poles inside the contour, so the integral around the contour is zero. When $x^0 - y^0 > 0$ and $M \to \infty$, the integral vanishes on S_M. We identify I_1 as a Cauchy principle value, which was defined in paper 1, so

$$I_1 = -\pi i \mathrm{Res}(q^0 = +|\mathbf{q}|) - \pi i \mathrm{Res}(q^0 = -|\mathbf{q}|) = \\ \frac{+\pi i}{2|\mathbf{q}|}\Big(\exp\left(-i|\mathbf{q}|(x^0 - y^0)\right) - \exp\left(+i|\mathbf{q}|(x^0 - y^0)\right)\Big) \quad (17)$$

subject to the condition that $x^0 - y^0 > 0$. Use this result in Eq. (14) and find

$$I = \left(\frac{+\pi i}{2|\mathbf{q}|} + C_1\right) \exp\left(-i|\mathbf{q}|(x^0 - y^0)\right) H(x^0 - y^0) + \\ \left(\frac{-\pi i}{2|\mathbf{q}|} + C_2\right) \exp\left(+i|\mathbf{q}|(x^0 - y^0)\right) H(x^0 - y^0). \quad (18)$$

H is the unit step function defined by $H(x^0 - y^0) = 1$ if $x^0 - y^0 > 0$ and $H(x^0 - y^0) = 0$ if $x^0 - y^0 < 0$. The first exponential represents a photon of positive energy traveling forward in time from y to x. The second term represents a photon of negative energy traveling forward in time. This is rejected as being unphysical. We can eliminate the second term by setting C_2 equal to $+\pi i/(2|\mathbf{q}|)$

Next, we evaluate Eq. (15) when $x^0 - y^0 < 0$. Introduce the closed contour:

(1) from $-M$ to $-|\mathbf{q}| - \epsilon$ along the q^0 axis

(2) from $-|\mathbf{q}| - \epsilon$ to $-|\mathbf{q}| + \epsilon$ along a semi-circle of radius ϵ, which is centered at $-|\mathbf{q}|$, and is above the q^0 axis.

(3) from $-|\mathbf{q}| + \epsilon$ to $+|\mathbf{q}| - \epsilon$ along the q^0 axis

(4) from $+|\mathbf{q}| - \epsilon$ to $+|\mathbf{q}| + \epsilon$ along a semi-circle of radius ϵ, which is centered at $+|\mathbf{q}|$, and is above the q^0 axis.

(5) from $+|\mathbf{q}| + \epsilon$ to $+M$ along the q^0 axis

(6) $+M$ to $-M$ along a semi-circle S_M of radius M, which is centered at the origin of the q^0 axis and is above the axis.

This contour will be referred to as contour C2. There are no poles inside the contour, so the integral around the contour is zero. When $x^0 - y^0 < 0$ and $M \to \infty$, the integral vanishes on S_M. Again, we identify I_1 as a Cauchy principle value, so

$$I_1 = +\pi i \mathrm{Res}(q^0 = +|\mathbf{q}|) + \pi i \mathrm{Res}(q^0 = -|\mathbf{q}|) = \frac{-\pi i}{2|\mathbf{q}|}\Big(\exp\left(-i|\mathbf{q}|(x^0 - y^0)\right) - \exp\left(+i|\mathbf{q}|(x^0 - y^0)\right)\Big). \quad (19)$$

Use this result in Eq. (14) and find

$$I = \Big(\frac{-\pi i}{2|\mathbf{q}|} + C_1\Big)\exp\left(-i|\mathbf{q}|(x^0 - y^0)\right)H(y^0 - x^0) + \Big(\frac{+\pi i}{2|\mathbf{q}|} + C_2\Big)\exp\left(+i|\mathbf{q}|(x^0 - y^0)\right)H(y^0 - x^0). \quad (20)$$

Use $x^0 - y^0 = -(y^0 - x^0)$ to rewrite Eq. (20)

$$I = \Big(\frac{-\pi i}{2|\mathbf{q}|} + C_1\Big)\exp\left(+i|\mathbf{q}|(y^0 - x^0)\right)H(y^0 - x^0) + \Big(\frac{+\pi i}{2|\mathbf{q}|} + C_2\Big)\exp\left(-i|\mathbf{q}|(y^0 - x^0)\right)H(y^0 - x^0). \quad (21)$$

The second exponential represents a photon of positive energy traveling forward in time from x to y. The first term represents a photon of negative energy traveling forward in time. This is rejected as being unphysical. We can eliminate the first term by setting C_1 equal to $+\pi i/(2|\mathbf{q}|)$. Thus Eq. (13) becomes

$$D_F(q) = -\Big[\frac{+1}{2|\mathbf{q}|}\Big(\frac{1}{q^0 - \mathbf{q}} - \pi i \delta(q^0 - |\mathbf{q}|)\Big) - \frac{1}{2|\mathbf{q}|}\Big(\frac{1}{q^0 + \mathbf{q}} + \pi i \delta(q^0 + \mathbf{q})\Big)\Big]. \quad (22)$$

We will find an alternate form for Eq. (22). Substitute Eq. (22) in Eq. (14). Integrate and find

$$I = \Big(\frac{+2\pi i}{2|\mathbf{q}|}\Big) \exp(-i|\mathbf{q}|(x^0 - y^0))H(x^0 - y^0) + \Big(\frac{+2\pi i}{2|\mathbf{q}|}\Big) \exp(+i|\mathbf{q}|(x^0 - y^0))H(y^0 - x^0). \quad (23)$$

Notice that we can recover this result if we move the pole at $q^0 = +|\mathbf{q}|$ below the q^0 axis by an amount ϵ, and if we move the negative pole at $q^0 = -|\mathbf{q}|$ above the q^0 axis by an amount ϵ. Thus

$$D_F(q) = -\Big[\frac{+1}{2|\mathbf{q}|}\Big(\frac{1}{q^0 - \mathbf{q} + i\epsilon}\Big) - \frac{1}{2|\mathbf{q}|}\Big(\frac{1}{q^0 + \mathbf{q} - i\epsilon}\Big)\Big]. \quad (24)$$

Notice that the pole at $q^0 = |\mathbf{q}| - i\epsilon$ is now enclosed by contour C1, and the pole at $q^0 = -|\mathbf{q}| + i\epsilon$ is now enclosed by contour C2. Thus, upon performing the integration in Eq. (14),

$$I = -2\pi i \mathrm{Res}(q^0 = |\mathbf{q}| - i\epsilon) + 2\pi i \mathrm{Res}(q^0 = |\mathbf{q}| - i\epsilon). \quad (25)$$

This result is identical to Eq. (23) when we let $\epsilon \to 0$ after integration. We shall often use Eq. (24) for $D_F(q)$.

Writing this equation over a common denominator, gives

$$D_F(q) = -\Big(\frac{1}{(q^0)^2 - |\mathbf{q}|^2 + i\epsilon}\Big). \quad (26)$$

The theory of generalized functions proves that

$$\frac{1}{x \pm i\epsilon} = \frac{1}{x} \mp \pi i \delta(x), \tag{27}$$

which also establishes the equality of Eq. (22)and Eq. (24).

The photon propagator will be used in the paper on Møller scattering. The same contour integration techniques are used in another review paper to find the Feynman propagator.