THE S-MATRIX (A REVIEW)

The Dirac equation for a free particle is

$$\left(i\hbar\gamma^\mu \frac{\partial}{\partial x^\mu} - mc\right)\phi = 0 \tag{1}$$

where for a positive energy electron

$$\phi(x) = \sqrt{\frac{mc^2}{EV}}\, u(p,s) \exp\left(-ip \cdot x/\hbar\right) \tag{2}$$

and u is a linear combination of w^1 and w^2, which were discussed in the review paper on the Dirac equation. Substitute ϕ in Eq. (1) and find

$$(\gamma^\mu p_\mu - mc)u = 0. \tag{3}$$

Take the adjoint of Eq. (3) and get

$$u^\dagger(\gamma^{\mu\dagger} p_\mu - mc) = u^\dagger(\gamma^0 p_0 - \gamma^j p_j - mc) = 0, \tag{4}$$

since $\gamma^{0\dagger} = \gamma^0$, and $\gamma^{j\dagger} = -\gamma^j$ for $j = 1, 2, 3$. As an example,

Date: July 25, 2012.

$$\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \text{ and } \gamma^{2\dagger} = \begin{pmatrix} 0 & 0 & 0 & +i \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ +i & 0 & 0 & 0 \end{pmatrix} = -\gamma^2.$$

The other identities follow similarly.

The Feynman free particle propagator is defined by the equation

$$i\hbar\gamma^\mu \frac{\partial S_F(x,y)}{\partial x^\mu} - mcS_F(x,y) = \delta^4(x-y). \tag{5}$$

The Dirac equation for a particle subjected to a vector potential, A, is

$$i\hbar\gamma^\mu \frac{\partial \psi}{\partial x^\mu} - mc\psi = +e\gamma^\mu A_\mu \psi. \tag{6}$$

The solution $\psi(x)$ is given by

$$\psi(x) = \phi(x) + e\int S_F(x,y)\gamma^\nu A_\nu(y)\psi(y)\, d^4y \tag{7}$$

where $S_F(x,y)$ is the Feynman propagator. It is verified that the above ψ is a solution to the Dirac equation by substituting Eq. (7) into the left hand side of Eq. (6). Thus

$$i\hbar\gamma^\mu \frac{\partial \psi}{\partial x^\mu} - mc\psi = i\hbar\gamma^\mu \frac{\partial \phi}{\partial x^\mu} - mc\phi + e\int \big(i\hbar\gamma^\mu \frac{\partial S_F(x,y)}{\partial x^\mu} - mcS_F(x,y)\big)\gamma^\nu A_\nu(y)\psi(y)\, d^4y \tag{8}$$

Use Eq. (1) and Eq. (5) in the above equation and get

$$i\hbar\gamma^{\mu}\frac{\partial\psi}{\partial x^{\mu}} - mc\psi = +e\int \delta^4(x-y)\gamma^{\nu}A_{\nu}(y)\psi(y)\,d^4y = e\gamma^{\nu}A_{\nu}(x)\psi(x), \tag{9}$$

which is identical to Eq. (6).

In a scattering experiment, we study the case where particles are prepared in the past(let $x^0 \to -\infty$) with a definite momentum p_i. The particles then interact with a vector potential. The resuting wave function ψ_i is given by Eq. (7). The probability amplitude that the particle emerges with momentum p_f as $x^0 \to +\infty$ is given by

$$S_{fi} = \lim_{x^0\to\infty}\int \phi_f^{\dagger}(x)\psi_i(x)d^3x \tag{10}$$

where S_{fi} is the S-matrix. Substitute Eq. (7) in Eq. (10) and find

$$S_{fi} = \lim_{x^0\to\infty}\Big(\int \phi_f^{\dagger}(x)\phi_i(x)d^3x + e\int \phi_f^{\dagger}(x)S_F(x,y)\gamma^{\nu}A_{\nu}(y)\psi_i(y)\,d^4y\,d^3x\Big). \tag{11}$$

Using box normalization, the first term integrates to

$$\lim_{x^0\to\infty}\left(\delta_{\mathbf{p}_f,\mathbf{p}_i}\exp\left(i(p_f^0 - p_i^0)x^0/\hbar\right)\right) = \delta_{\mathbf{p}_f,\mathbf{p}_i} \tag{12}$$

Write the second term as

$$e\int I(x^0,y)\gamma^{\nu}A_{\nu}(y)\psi_i(y)d^4y \tag{13}$$

where

$$I(x^0, y) = \int \phi_f^\dagger(x) S_F(x,y) d^3x. \qquad (14)$$

Writing $S_F(x,y)$ as The Fourier transform of $S_F(k)$,

$$I(x^0, y) = \sqrt{\frac{mc^2}{E_f V}} \int u_f^\dagger \exp{(ip_f \cdot x/\hbar)} \exp{(-ik \cdot (x-y))} S_F(k) \frac{d^4k}{(2\pi)^4} d^3x. \qquad (15)$$

$$I(x^0, y) = \sqrt{\frac{mc^2}{E_f V}} \int u_f^\dagger \exp{(-ix \cdot (k - p_f/\hbar))} \exp{(+ik \cdot y)} S_F(k) \frac{d^4k}{(2\pi)^4} d^3x. \qquad (16)$$

$$I(x^0, y) = \sqrt{\frac{mc^2}{E_f V}} \int u_f^\dagger (2\pi)^3 \delta^3(\mathbf{p_f}/\hbar - \mathbf{k}) \exp{(-ix^0(k^0 - p_f^0/\hbar))} \exp{(+ik \cdot y)} S_F(k) \frac{d^4k}{(2\pi)^4}. \qquad (17)$$

Recall from the paper on the Feynmam propagator that

$$S_F(k) = \frac{(\hbar\gamma^\mu k_\mu + mc)}{2\sqrt{\hbar^2|\mathbf{k}|^2 + m^2c^2}} \Big(\frac{1}{\hbar k^0 - \sqrt{\hbar^2|\mathbf{k}|^2 + m^2c^2} + i\epsilon} - \frac{1}{\hbar k^0 + \sqrt{\hbar^2|\mathbf{k}|^2 + m^2c^2} - i\epsilon} \Big) \qquad (18)$$

$$I(x^0, y) = \sqrt{\frac{mc^2}{E_f V}} \int u_f^\dagger \exp\left(+ix^0 p_f^0/\hbar - i\mathbf{p_f}\cdot\mathbf{y}/\hbar\right) \exp\left(-ik^0(x^0 - y^0)\right)$$
$$\frac{(\hbar\gamma^0 k_0 + \gamma^j p_{jf} + mc)}{2\sqrt{|\mathbf{p}_f|^2 + m^2c^2}} \Big(\frac{1}{\hbar k^0 - \sqrt{|\mathbf{p}_f|^2 + m^2c^2} + i\epsilon}$$
$$- \frac{1}{\hbar k^0 + \sqrt{|\mathbf{p}_f|^2 + m^2c^2} - i\epsilon} \Big) \frac{dk^0}{(2\pi)}. \quad (19)$$

Perform a contour integration and find

$$I(x^0, y) = \sqrt{\frac{mc^2}{E_f V}} \Big(\frac{-2\pi i}{2\pi} \frac{H(x^0 - y^0)}{2\sqrt{|\mathbf{p}_f|^2 + m^2c^2}} u^\dagger (\gamma^0 \sqrt{|\mathbf{p}_f|^2 + m^2c^2} + \gamma^j p_{jf} + mc\Big)$$
$$\exp\left(ix^0 p_f^0/\hbar - i\mathbf{p_f}\cdot\mathbf{y}/\hbar\right) exp\Big(-i\sqrt{|\mathbf{p}_f|^2 + m^2c^2}(x^0 - y^0)/\hbar\Big)$$
$$+ \frac{-2\pi i}{2\pi} \frac{H(y^0 - x^0)}{2\sqrt{|\mathbf{p}_f|^2 + m^2c^2}} u^\dagger \Big(-\gamma^0 \sqrt{|\mathbf{p}_f|^2 + m^2c^2} + \gamma^j p_{jf} + mc\Big)$$
$$\exp\left(ix^0 p_f^0/\hbar - i\mathbf{p_f}\cdot\mathbf{y}/\hbar\right) exp\Big(+i\sqrt{|\mathbf{p}_f|^2 + m^2c^2}(x^0 - y^0)/\hbar\Big)\Big), \quad (20)$$

Since $p_f^0 = \sqrt{|\mathbf{p}_f|^2 + m^2c^2}$, the second term in Eq. (20) vanishes by Eq. (4). Add and subtract $\gamma^0 \sqrt{|\mathbf{p}_f|^2 + m^2c^2}$ within the parentheses of the remaining term, use Eq. (4) again, and find

$$u^\dagger (+\gamma^0 \sqrt{|\mathbf{p}_f|^2 + m^2c^2} + \gamma^j p_{jf} + mc) =$$
$$u^\dagger (+2\gamma^0 \sqrt{|\mathbf{p}_f|^2 + m^2c^2} - \gamma^0 \sqrt{|\mathbf{p}_f|^2 + m^2c^2} + \gamma^j p_{jf} + mc) =$$
$$2u^\dagger \gamma^0 \sqrt{|\mathbf{p}_f|^2 + m^2c^2} \quad (21)$$

So finally

$$I(x^0, y) = \sqrt{\frac{mc^2}{E_f V}}\, \bar{u} \exp\left(ip \cdot y/\hbar\right) H(x_0 - y_0) = \bar{\phi}(y) H(x_0 - y_0) \quad (22)$$

where by definition, $\bar{u} = u^\dagger \gamma^0$ and $\bar{\phi} = \phi^\dagger \gamma^0$. Substitute Eq. (22) into Eq. (11) and find

$$S_{fi} = \delta_{\mathbf{p}_f, \mathbf{p}_i} + e \int \bar{\phi}_f(y) \gamma^\nu A_\nu(y) \psi_i(y)\, d^4 y. \quad (23)$$

The S-matrix is approximated by replacing ψ_i by ϕ_i. Then

$$S_{fi} = \delta_{\mathbf{p}_f, \mathbf{p}_i} + e \int \bar{\phi}_f(y) \gamma^\nu A_\nu(y) \phi_i(y)\, d^4 y. \quad (24)$$

This is the formula for the S-matrix that is used in the paper on Mott Rutherford scattering.