ELECTRON SCATTERING BY NUCLEI

Abstract

The electron charge is considered to be distributed or extended in space. The differential of the electron charge is set equal to a function of the electron charge coordinates multiplied by a four-dimensional differential volume element. The four-dimensional integral of this function is required to equal the electron charge in all Lorentz frames. The S-matrix for the scattering of such an electron by the Coulomb potential of a nucleus is calculated. This modification is related to replacing the classical potential energy of a point electron in a Coulomb potential by the classical potential energy of an extended electron in a Coulomb potential. The result is that the S-matrix of the extended electron theory is a product of the S-matrix of the point electron scattering by a nucleus times an expression which is dependent on the electron size and structure and which is called the electron form factor. Thus the S-matrix contains a nuclear form factor and an electron form factor.

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I. INTRODUCTION

In the rest frame of an electron charge distribution, let $x_r'^{\mu} = (x_r'^{0}, x_r'^{1}, x_r'^{2}, x_r'^{3})$ denote a spacetime charge point, and let $x_r^{\mu} = (x_r^0, x_r^1, x_r^2, x_r^3)$ denote the center of the charge distribution. The charge distribution of the electron is assumed to have a well-defined center, which is identified as the argument of the wave function. The shape of the charge distribution depends on the motion of the charge, and is assumed to be unaffected by any interaction. Sometimes the superscript on the four-vector (not the components) will be omitted, and we will write $x_r' = (x_r'^{0}, x_r'^{1}, x_r'^{2}, x_r'^{3})$ and $x_r = (x_r^0, x_r^1, x_r^2, x_r^3)$. Introduce $\tilde{x}_r = x_r' - x_r$ or equivalently $\tilde{x}_r^{\mu} = x_r'^{\mu} - x_r^{\mu}$. In a frame of reference in which the electron charge distribution moves with a speed β in the $+x^3$ direction, let $x_m' = (x_m'^{0}, x_m'^{1}, x_m'^{2}, x_m'^{3})$ denote a spacetime charge point, and let $x_m = (x_m^0, x_m^1, x_m^2, x_m^3)$ denote the center of the charge distribution. Introduce $\tilde{x}_m = x_m' - x_m$. A Lorentz transformation yields $\tilde{x}_r^1 = \tilde{x}_m^1$, $\tilde{x}_r^2 = \tilde{x}_m^2$, $\tilde{x}_r^3 = \gamma(\tilde{x}_m^3 - \beta\tilde{x}_m^0)$, and $\tilde{x}_r^0 = \gamma(\tilde{x}_m^0 - \beta\tilde{x}_m^3)$ where $\gamma = 1/\sqrt{1-\beta^2}$. Denote this Lorentz transformation by $\tilde{x}_r^{\mu} = L(\tilde{x}_m^{\mu})$.

In the rest frame, the electron charge e is equal to $\int \rho_r(\tilde{x}_r)\delta(\tilde{x}_r^0)d^4\tilde{x}_r$ where $\rho_r(\tilde{x}_r)$ is the charge density in the rest frame and δ denotes the delta function.[1] In the m frame, the electric charge e is equal to

$\int \rho_r(L(\tilde{x}_m))\delta[\gamma(\tilde{x}^0_m - \beta\tilde{x}^3_m)]d^4\tilde{x}_m$.[2] So an element of charge de_m in the m frame is given by

$$de_m = \rho_r(L(\tilde{x}_m))\delta[\gamma(\tilde{x}^0_m - \beta\tilde{x}^3_m)]d^4\tilde{x}_m. \tag{1}$$

The next section reviews the calculation of the S-matrix for point electron scattering in the Coulomb potential of a nucleus. The third section treats extended electron scattering by a nucleus. The result is that the new S-matrix is the S-matrix for point electron scattering by a nucleus times an electron form factor. Thus the S-matrix contains both a nuclear form factor and an electron form factor. The paper ends with a short discussion.

II. ELECTRON-NUCLEON SCATTERING

The calculation of the S matrix for electron scattering from a fixed Coulomb potential will follow Bjorken and Drell[3]. For the point electron, the S matrix element is approximated by

$$S_{fi} = -i \int d^4x\, \bar{\phi}_f(x)(e\gamma^0)A_0(\mathbf{x})\phi_i(x). \tag{2}$$

The initial exact wave function is replaced by the plane wave solution to the Dirac equation. The plane wave solution, which is normalized to unity in a box of volume V, is

$$\phi_i(x) = \sqrt{\frac{m}{E_i V}}\, u_i \exp{(-ip_i \cdot x)}, \tag{3}$$

where $\hbar$ and c have been set equal to 1, m is the electron rest mass, $p_i^\mu = (E_i, p_i^1, p_i^2, p_i^3)$ is its initial four-momentum, u_i is a four-component spinor, which depends on the initial four-momentum and the initial spin, and the Dirac matrix $\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Similarly, the final electron wave function is

$$\phi_f(x) = \sqrt{\frac{m}{E_f V}}\, u_f \exp{(-ip_f \cdot x)} \tag{4}$$

where p_f is the final four-momentum, E_f is the final energy, u_f is the final four-component spinor, and $\bar{\phi}_f = \phi_f^\dagger \gamma^0$. The potential at an electron charge point x, which is due to the nucleus at rest at the origin, is given by

$$A_0(\mathbf{x}) = \int \frac{d^3y'_r\, \rho_n(\mathbf{y}'_r)}{4\pi|\mathbf{x} - \mathbf{y}'_r|}\,. \tag{5}$$

Here ρ_n is the nuclear charge density, the nuclear charge coordinates are $\mathbf{y}'_r = (y'^1_r, y'^2_r, y'^3_r)$, and the nuclear charge $-Ze$ is given by

$$-Ze = \int \rho_n(\mathbf{y}'_r) d^3 y'_r, \tag{6}$$

Substitution of Eqs. (3), (4), and (5) into Eq. (2) yields

$$S_{fi} = \frac{-iem}{V\sqrt{E_i E_f}} \bar{u}_f \gamma^0 u_i \int \frac{\exp\left[i(p_f - p_i)\cdot x\right] \rho_n(\mathbf{y}'_r) d^3 y'_r \, d^4 x}{4\pi |\mathbf{x} - \mathbf{y}'_r|} \tag{7}$$

Set $\mathbf{z} = \mathbf{x} - \mathbf{y'_r}$, $q = p_f - p_i$, and change integration variables from $\mathbf{x}$ to $\mathbf{z}$. Then

$$S_{fi} = \frac{-iem}{V\sqrt{E_i E_f}} \bar{u}_f \gamma^0 u_i \int dx^0 \exp\left(iq^0 x^0\right) \int \frac{\exp\left[-i\mathbf{q}\cdot(\mathbf{z} + \mathbf{y}'_r)\right] \rho_n(\mathbf{y}'_r) d^3 y'_r \, d^3 z}{4\pi |\mathbf{z}|} \tag{8}$$

Perform the following integrations:

$$\int dx^0 \exp\left(iq^0 x^0\right) = 2\pi\delta(q^0) = 2\pi\delta(E_f - E_i); \tag{9}$$

$$\int \frac{\exp\left(-i\mathbf{q}\cdot\mathbf{z}\right)}{|\mathbf{z}|} d^3 z = \frac{4\pi}{|\mathbf{q}|^2} = \frac{4\pi}{|\mathbf{p_f} - \mathbf{p_i}|^2};^4 \tag{10}$$

and find

$$S_{fi} = \frac{iZe^2 m}{V\sqrt{E_i E_f}} \frac{\bar{u}_f \, \gamma^0 u_i}{|\mathbf{q}|^2} 2\pi\delta(E_f - E_i) F_n(q) \tag{11}$$

where $F_n(q)$, the nuclear form factor, is given by

$$F_n(q) = \frac{1}{-Ze}\int \rho_n(\mathbf{y}'_r)\exp\left(-i\mathbf{q}\cdot\mathbf{y}'_r\right)d^3y'_r\,. \tag{12}$$

See the review paper by Hofstadter for more on the nuclear form factor.[5]

III. EXTENDED ELECTRON SCATTERING FROM A NUCLEUS

In the previous section, x was the argument of the wave function, and also the electron spacetime charge point in an arbitrary Lorentz frame. Take the electron to be initially moving with a speed β in the $+x^3$ direction. This is the m frame. So now x_m is the argument of the wave function, and also the center of the electron charge.

Eq. (2) will be modified to take into account the spatial distribution of the electron charge. The interaction takes place at the charge points x'_m, so Eq. (5) is replaced by

$$A_0(\mathbf{x}'_m) = \int \frac{d^3y'_r\,\rho_n(\mathbf{y}'_r)}{4\pi|\mathbf{x}'_m - \mathbf{y}'_r|} = \int \frac{d^4y'_r\,\rho_n(\mathbf{y}'_r)\delta(y'^{\,0}_r)}{4\pi|\mathbf{x}'_m - \mathbf{y}'_r|}. \tag{13}$$

Use has been made of $\int \delta(y'^{\,0}_r)dy'^{\,0}_r = 1$. Replace the electron charge by the four dimensional integral of $\rho_r(L(\tilde{x}_m))\delta[\gamma(\tilde{x}^0_m - \beta\tilde{x}^3_m)]d^4\tilde{x}_m$. Making these substitutions in Eq. (2), the S matrix for the extended electron is

$$S_{FI} = -i \int d^4x_m \, \bar{\phi}_f(x_m) \rho_r(L(\tilde{x}_m)) \delta[\gamma(\tilde{x}^0_m - \beta \tilde{x}^3_m)] d^4\tilde{x}_m \, (\gamma^0) \frac{d^4y'_r \, \rho_n(\mathbf{y}'_r)) \delta({y'_r}^0)}{(4\pi|\mathbf{x}'_m - \mathbf{y}'_r|)} \phi_i(x_m). \quad (14)$$

After substituting Eqs. (3) and (4) into Eq. (14),

$$S_{FI} = \frac{iZe^2m}{V\sqrt{E_iE_f}} \, \bar{u}_f \gamma^0 u_i \int \exp{(i(p^0_f - p^0_i)x^0_m)} dx^0_m \int \frac{d^3x_m \exp{[-i(\mathbf{p}_f - \mathbf{p}_i) \cdot \mathbf{x}_m]}}{4\pi|\mathbf{x}'_m - \mathbf{y}'_r|} \frac{\rho_n(\mathbf{y}'_r) \delta({y'_r}^0) d^4y'_r}{-Ze} \frac{\rho_r(L(\tilde{x}_m))}{e} \, \delta[\gamma(\tilde{x}^0_m - \beta \tilde{x}^3_m)] d^4\tilde{x}_m. \quad (15)$$

Now p_f and p_i refer to momenta in the m frame, so set $q_m = p_f - p_i$.

Introduce $z = x'_m - y'_r = x_m + \tilde{x}_m - y'_r$. Change variables from x_m to z.

Then

$$S_{FI} = \frac{iZe^2m}{V\sqrt{E_iE_f}} \, (\bar{u}_f \gamma^0 u_i) \int \exp{(iq^0_m \, z^0)} dz^0 \int \frac{d^3z \exp{(-i\mathbf{q_m} \cdot \mathbf{z})}}{4\pi|\mathbf{z}|} \int \frac{\rho_n(\mathbf{y}'_r) \delta({y'_r}^0) \exp{(+iq_m \cdot y'_r)} d^4y'_r}{-Ze} \int \exp{(-iq_m \cdot \tilde{x}_m)} \frac{\rho_r(L(\tilde{x}_m))}{e} \, \delta[\gamma(\tilde{x}^0_m - \beta \tilde{x}^3_m)] d^4\tilde{x}_m \quad (16)$$

By Eqns. (9), (10), (12), and (14)

$$S_{FI} = S_{fi}\,F(q) \tag{17}$$

where $F(q)$, the electron form factor, is given by

$$F(q) = \frac{1}{e}\int \exp\left(-iq_m \cdot \tilde{x}_m\right)\rho_r(L(\tilde{x}_m))\delta[\gamma(\tilde{x}_m^0 - \beta\tilde{x}_m^3)]d^4\tilde{x}_m. \tag{18}$$

It has been shown that the electron form factor is invariant,[2] and that

$$F(q) = \frac{1}{e}\int \exp\left(-iq_r \cdot \tilde{x}_r\right)\rho_r(\tilde{x}_r))\delta(x_r^0)d^4\tilde{x}_r \tag{19}$$

where $q_r = L(q_m) = (\gamma_i(q_m^0 - \beta_i q_m^3), q_m^1, q_m^2, \gamma_i(q_m^3 - \beta_i q_m^0))$ is the q vector in the rest frame, but expressed as the q vector in the m frame. In the same paper, it has been shown

$$|\mathbf{q}_r|^2 = \frac{(p_f \cdot p_i)^2 - m^4}{m^2} = \frac{4|\mathbf{p_i}^2|\sin^2(\theta/2)(1 - \beta^2\cos^2(\theta/2))}{(1 - \beta^2)}. \tag{20}$$

where θ is the scattering angle.

To get a rough idea of how electron size affects scattering, pick the electron charge density to be given by

$$\rho_r(\tilde{x}_r) = \frac{e}{4\pi a^2}\delta\left(\sqrt{(\tilde{x}_r^1)^2 + (\tilde{x}_r^2)^2 + (\tilde{x}_r^3)^2} - a\right). \tag{21}$$

Then in spherical coordinates

$$F(q) = \int \exp\left(i\tilde{r}_r|\mathbf{q}_r|\cos\tilde{\theta}\right)\frac{\delta(\tilde{r}_r - a)(\tilde{r}_r)^2 \sin\tilde{\theta}_r\, d\tilde{\theta}_r\, d\tilde{\phi}_r\, d\tilde{r}_r}{4\pi a^2} \tag{22}$$

where $|\mathbf{q}_r|^2 = (q^1)^2 + (q^2)^2 + \gamma^2(q^0 - \beta\, q^3)^2$. After integration,

$$F(q) = \frac{\exp\left(i|\mathbf{q}_r|a\right) - \exp\left(-i|\mathbf{q}_r|a\right)}{2i|\mathbf{q}_r|a} = \frac{\sin\left(|\mathbf{q}_r|a\right)}{|\mathbf{q}_r|a} = j_0(|\mathbf{q}_r|a) \tag{23}$$

where j_0 is the spherical Bessel function of order zero.

IV. DISCUSSION

The purpose of the paper is to investigate how electron size and structure affect electron scattering. It was found that the S-matrix of the extended electron theory is the S-matrix of the point electron theory multiplied by the electron form factor.

Eq. (21) was chosen for the charge density because of the simple picture it presents of an extended charge and also because of its mathematical simplicity. This choice gives a rough idea of the effect of size and structure on scattering.

The S-matrix of the point electron theory contains products of probability amplitudes, e.g., it contains the probability amplitude for the interaction of the electron with the Coulomb field to occur at x, the electron charge point and the argument of the wave function. In the extended electron theory, the S-matrix contains an integral over the probability amplitudes that the electron interacts with the potential at the electron charge points x' and not x. This way of thinking can replace the classical picture of the electron as an extended charge.

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