ELECTRON-MUON SCATTERING

ABSTRACT

The electron charge is considered to be distributed or extended in space. The differential of the electron charge is set equal to a function of electron charge coordinates multiplied by a four-dimensional differential volume element. The four-dimensional integral of this function is required to equal the electron charge in all Lorentz frames. The muon is treated similarly. The S-matrix for extended electron-extended muon scattering is calculated. The result is that the S-matrix of the extended electron theory is a product of the S-matrix of the point electron theory multiplied by an electron form factor and a muon form factor.

I. INTRODUCTION

In the rest frame of an electron charge distribution, let $x_r'^{\nu} = (x_r'^0, x_r'^1, x_r'^2, x_r'^3)$ denote a spacetime charge point, and let $x_r^{\nu} = (x_r^0, x_r^1, x_r^2, x_r^3)$ denote the center of the charge distribution. Sometimes the superscript on the four-vector (not the components) will be omitted, and we will write

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$x'_r = (x'^0_r, x'^1_r, x'^2_r, x'^3_r)$ and $x_r = (x^0_r, x^1_r, x^2_r, x^3_r)$. Introduce $\tilde{x}_r = x'_r - x_r$ or equivalently $\tilde{x}^\nu_r = x'^\nu_r - x^\nu_r$. In a frame of reference in which the electron charge distribution moves with a speed β in the $+x^3$ direction, let $x'_m = (x'^0_m, x'^1_m, x'^2_m, x'^3_m)$ denote a spacetime charge point, and let $x_m = (x^0_m, x^1_m, x^2_m, x^3_m)$ denote the center of the charge distribution. Introduce $\tilde{x}_m = x'_m - x_m$. A Lorentz transformation yields $\tilde{x}^1_r = \tilde{x}^1_m$, $\tilde{x}^2_r = \tilde{x}^2_m$, $\tilde{x}^3_r = \gamma(\tilde{x}^3_m - \beta\tilde{x}^0_m)$, and $\tilde{x}^0_r = \gamma(\tilde{x}^0_m - \beta\tilde{x}^3_m)$ where $\gamma = 1/\sqrt{1-\beta^2}$. Denote this Lorentz transformation by $\tilde{x}_r = L(\tilde{x}_m)$.

In the rest frame, the electron charge e is equal to $\int \rho_r(\tilde{x}_r)\delta(\tilde{x}^0_r)d^4\tilde{x}_r$ where $\rho_r(\tilde{x}_r)$ is the charge density in the rest frame and δ denotes the delta function.[1] In the m frame, the electric charge e is equal to $\int \rho_r(L(\tilde{x}_m))\delta[\gamma(\tilde{x}^0_m - \beta\tilde{x}^3_m)]d^4\tilde{x}_m$.[2] So an element of charge de_m in the m frame is given by

$$de_m = \rho_r(L(\tilde{x}_m))\delta[\gamma(\tilde{x}^0_m - \beta\tilde{x}^3_m)]d^4\tilde{x}_m. \tag{1}$$

The next section is a review of point electron scattering by a point muon. The third section will study scattering of an extended electron by an extended muon. The S-matrix of the extended electron theory is found to be the product of the S-matrix of the point electron theory multiplied by an electron form factor and multiplied by a muon form factor. The fourth section will introduce a new convention for keeping track of the imaginary i's. A short discussion follows.

II. ELECTRON-MUON SCATTERING

The calculation for electron-muon scattering will follow the calculation of electron-proton scattering in Bjorken and Drell where the proton is treated as a point particle of spin 1/2.[3] For the point electron, the S matrix element is approximated by

$$S_{fi} = -i \int d^4x \, \bar{\phi}_f(x)(e\gamma^\nu)A_\nu(x)\phi_i(x). \tag{2}$$

This equation can be used to study electron-muon scattering since the muon is treated here as a point particle of spin 1/2. The initial exact electron wave function is approximated by the plane wave solution to the Dirac equation. The plane wave solution, which is normalized to unity in a box of volume V, is

$$\phi_i(x) = \sqrt{\frac{m}{E_i V}}\, u_i \exp\left(-ip_i \cdot x\right) \tag{3}$$

where $\hbar$ and c have been set equal to 1, m is the electron rest mass, u_i is a four-component spinor, which depends on the initial spin and on $p_i = (p_i^0 = E_i, p_i^1, p_i^2, p_i^3)$, the initial four-momentum, and γ^ν are the four Dirac matrices, which are labelled by $\nu = 0, 1, 2, 3$. The final electron wave function is

$$\phi_f(x) = \sqrt{\frac{m}{E_f V}}\, u_f \exp\left(-ip_f \cdot x\right), \tag{4}$$

where p_f is the final electron four-momentum, E_f is the final electron energy, u_f is the final electron spinor, and $\bar{\phi}_f = \phi_f^\dagger \gamma^0$. The vector potential is given by

$$A_\nu(x) = \int d^4y\, D_F(x-y)\, J_\nu(y) \tag{5}$$

where $D_F(x-y)$ is the photon propagator, and $J_\nu(y)$ is the muon current. The photon propagator is

$$D_F(x-y) = \int \frac{d^4q}{(2\pi)^4} \exp\left[-iq \cdot (x-y)\right] \frac{-1}{q^2 + i\epsilon} \tag{6}$$

where q is the four-momentum of the photon. For a negatively charged muon, the muon current is identified as $\phi_F(y) e \gamma_\nu\, \phi_F(y)$ where $\phi_F(y)$ is the final muon wave function and $\phi_I(y)$ is the initial muon wave function. The initial muon wave function is approximated by the plane wave

$$\phi_I(y) = \sqrt{\frac{M}{E_I V}}\, u_I \exp(-ip_I \cdot y) \tag{7}$$

where M is the muon mass, u_I is a four component spinor, which depends on the initial muon spin and on $p_I = (p_I^0 = E_I, p_I^1, p_I^2, p_I^3)$, the initial muon four-momentum. The final muon wave function is

$$\phi_F(y) = \sqrt{\frac{M}{E_F V}}\, u_F \exp(-ip_F \cdot y) \tag{8}$$

where p_F is the final muon momentum four-vector,E_F is the final muon energy, and u_F is the final four-component spinor of the muon. Thus, the S-matrix for electron-muon scattering is

$$S_{fi} = -i \int d^4x\, d^4y\, \bar{\phi}_f(x)(e\gamma^\nu)\phi_i(x) D_F(x-y)\bar{\phi}_F(y)(e\gamma_\nu)\phi_I(y). \tag{9}$$

Substituting Eqs. (3), (4), (6), (7), and (8) into Eq. (9) yields

$$S_{fi} = \frac{+ie^2 mM(\bar{u}_f\gamma^\nu u_i)(\bar{u}_F\gamma_\nu u_I)}{(2\pi)^4 V^2\sqrt{E_i E_f E_I E_F}} \int \frac{d^4x\, d^4y\, d^4q\, \exp[i(p_f - p_i - q)\cdot x]\exp[i(p_F - p_I + q)\cdot y]}{q^2 + i\epsilon}. \tag{10}$$

Perform the following integrations:

$$\int \exp(i(p_f - p_i - q)\cdot x) d^4x = (2\pi)^4 \delta^4(p_f - p_i - q); \tag{11}$$

$$\int \exp(i(p_F - p_I + q) \cdot y) d^4y = (2\pi)^4 \delta^4(p_F - p_I + q); \qquad (12)$$

$$\int \delta^4(p_f - p_i - q)\delta^4(p_F - p_I + q)\frac{d^4q}{q^2 + i\epsilon} = \frac{\delta^4(p_F + p_f - p_I - p_i)}{(p_f - p_i)^2}; \qquad (13)$$

and find

$$S_{fi} = \frac{+ie^2mM}{V^2\sqrt{E_iE_fE_IE_F}}(2\pi)^4\delta^4(p_f + p_F - p_i - p_I)\frac{(\bar{u}_f\gamma^\nu u_i)(\bar{u}_F\gamma_\nu u_I)}{(p_f - p_i)^2}. \qquad (14)$$

III. EXTENDED ELECTRON-MUON SCATTERING

In the previous section, x was the argument of the electron wave function and the electron charge spacetime point in an arbitrary Lorentz frame. Take the electron to be initally moving with a speed β in the $+x^3$ direction. This previously was called the m frame. So now x_m is the argument of the wave function and also the center of the electron charge distribution.

Suppose that the muon is initially and unrealistically at rest in the m frame. Then, let $y'_r = (y'^0_r, y'^1_r, y'^2_r, y'^3_r)$ denote a spacetime muon charge point, and let $y_r = (y^0_r, y^1_r, y^2_r, y^3_r)$ denote the center of the muon charge distribution in the m frame and also the argument of the muon wave function. Introduce $\tilde{y}_r = y'_r - y_r$. The subscript r is attached to y, y', and $\tilde{y}$ to emphasize that the muon is at rest in the m frame.

Eq. (9) will be modified to take into account the spatial charge distribution of the electron and also the muon. The interaction takes place at charge points, so replace $D_F(x-y)$ by $D_F(x'_m - y'_r)$. In addition, the electron charge is replaced by the four-dimensional integral of de_m where de_m is given by Eq. (1). Since the muon is at rest in the m frame, replace the muon charge by the four-dimensional integral of

$$de_{\mu r} = \rho_{\mu r}(\tilde{y}_r))\delta(\tilde{y}^0_r)d^4\tilde{y}_r. \tag{15}$$

Here $\rho_{\mu r}$ is the muon charge density in the rest frame of the muon.

For the extended electron and the extended muon scattering in the m frame, the S-matrix is

$$S_{FI} = -i\int d^4x_m\, d^4y_r\, \bar{\phi}_f(x_m)(de_m)\gamma^\nu\phi_i(x_m)$$
$$D_F(x'_m - y'_r)\bar{\phi}_F(y_r)(de_{\mu r})\gamma_\nu\phi_I(y_r). \tag{16}$$

So now the S-matrix is

$$S_{FI} = -i \int d^4x_m \, d^4y_m \, \bar{\phi}_f(x_m)(\gamma^\nu)\phi_i(x_m)\rho_r(L(\tilde{x}_m))\delta[\gamma(\tilde{x}^0_m - \beta\tilde{x}^3_m)]d^4\tilde{x}_m$$

$$D_F(x'_m - y'_r)\bar{\phi}_F(y_r)(\gamma_\nu)\phi_I(y_r)\rho_{\mu r}(\tilde{y}_r)\delta(\tilde{y}^0_r)d^4\tilde{y}_r. \quad (17)$$

Use $D_F(x'_m - y'_r) = D_F(x_m - y_r)\ \exp(-i\tilde{x}_m \cdot q_m)\ \exp(+i\tilde{y}_r \cdot q_m)$ to show

$$S_{FI} = -i \int d^4x_m \, d^4y_r \, \bar{\phi}_f(x_m)(e\gamma^\nu)\phi_i(x_m)D_F(x_m - y_r)\bar{\phi}_F(y_r)(e\gamma_\nu)\phi_I(y_r)$$

$$\int \exp(-i\tilde{x}_m \cdot q_m)\frac{\rho_r(L(\tilde{x}_m))}{e}\delta[\gamma(\tilde{x}^0_m - \beta\tilde{x}^3_m)]d^4\tilde{x}_m$$

$$\int \exp(+i\tilde{y}_r \cdot q_m)\frac{\rho_{\mu r}(\tilde{y}_r))}{e}\delta(\tilde{y}^0_r)d^4\tilde{y}_r\,. \quad (18)$$

The first integral is identified as S_{fi} (see Eq. (9) and Eq. (14)). By Eq. (11) and Eq. (12), $q_m = p_f - p_i = p_I - p_F$ where the momenta are measured in the m frame. Finally,

$$S_{FI} = S_{fi}F(q)F_\mu(q) \quad (19)$$

where the electron form factor

$$F(q) = \int \exp(-i\tilde{x}_m \cdot q_m)\frac{\rho_r(L(\tilde{x}_m))}{e}\delta[\gamma(\tilde{x}^0_m - \beta\tilde{x}^3_m)]d^4\tilde{x}_m\,, \quad (20)$$

and the muon form factor

$$F_\mu(q) = \int \exp(+i\tilde{y}_r \cdot q_m)\frac{\rho_{\mu r}(\tilde{y}_r))}{e}\delta(\tilde{y}_r^0)d^4\tilde{y}_r\,. \tag{21}$$

Thus the S-matrix for extended electron and extended muon scattering is the S-matrix for point electron-muon scattering times an electron form factor times a muon form factor. As previously shown,[2] the form factor is invariant, so

$$F(q) = \int \exp(-i\tilde{x}_r \cdot q_r)\frac{\rho_r(\tilde{x}_r)}{e}\,\delta(\tilde{x}_r^0)d^4\tilde{x}_r\,. \tag{22}$$

To get a rough idea of how size and structure affect scattering, pick the electron charge to be uniformly distributed on spherical shell of radius a in the rest frame. So

$$\rho_r(\tilde{x}_r) = \frac{e}{4\pi a^2}\,\delta(\sqrt{(\tilde{x}_r^1)^2 + (\tilde{x}_r^2)^2 + (\tilde{x}_r)^2} - a). \tag{23}$$

In spherical coordinates, $(\tilde{r}_r)^2 = (\tilde{x}_r^1)^2 + (\tilde{x}_r^2)^2 + (\tilde{x}_r^3)^2$, so that $d^3\tilde{x}_r = (\tilde{r}_r)^2 \sin\tilde{\theta}_r\, d\tilde{\theta}_r\, d\tilde{\phi}_r\, d\tilde{r}_r$. Then

$$F(q) = \int \frac{\delta(|\tilde{\mathbf{r}}| - a)}{4\pi a^2}\exp\left(+i\mathbf{q}_r \cdot \tilde{\mathbf{r}}\right)\hat{r}^2 \sin\tilde{\theta}\, d\tilde{\theta}\, d\tilde{\phi}\, d\tilde{r} = j_0(|\mathbf{q}_r|a) \tag{24}$$

where $|\mathbf{q}_r|^2 = (q_r^1)^2 + (q_r^2)^2 + (q_r^3)^2 = (q_m^1)^2 + (q_m^2)^2 + \gamma^2(q_m^3 - \beta q_m^0)^2$ and j_0 is the spherical Bessel function of order 0. Here $|\mathbf{q}_r|$ is expressed in terms of the components of $q_m = p_f - p_i$, since it is p_f and p_i which are measured in the m or lab frame.

Similarly, pick the muon charge distribution to be a a spherical shell of radius a_μ in the rest frame. Then,

$$\rho_{r\mu}(\tilde{y}_r) = \frac{e}{4\pi a_\mu^2}\,\delta(\sqrt{(\tilde{y}_r^1)^2 + (\tilde{y}_r^2)^2 + (\tilde{y}_r^3)^2} - a_\mu). \tag{25}$$

Introduce spherical coordinates again. Proceed similarly with Eq. (21), and find $j_0(|\mathbf{q}_m|a_\mu)$ where $|\mathbf{q}_m| = |\mathbf{p}_f - \mathbf{p}_i| = \sqrt{|\mathbf{p}_f|^2 + |\mathbf{p}_i|^2 - 2\mathbf{p}_f \cdot \mathbf{p}_i}$.

IV. NEW CONVENTION

The following convention has been adopted to keep track of the imaginary i's in more complicated situations: multiply each charge e by $-i$; and multiply the photon propagator by i.[4] Thus, Eq. (9) becomes

$$S_{fi} = \int d^4x\, d^4y\, \bar{\phi}_f(x)(-ie\gamma^\mu)\phi_i(x) iD_F(x-y)\bar{\phi}_F(y)(-ie\gamma_\mu)\phi_I(y). \tag{26}$$

V. DISCUSION

The result is that the S-matrix of the extended particle theory is the product of the S-matrix of the point particle theory multiplied by form factors of each of the extended particles. The charge densities used in section III. were chosen for their mathematical simplicity. The charge densities should be calculated from the experimentally determined form factors. On the other hand, if there are no experimental deviations from the point electron theory, this sets an upper limit on the radius of the extended particles. See the soon to be posted paper on Møller scattering for estimates of the electron radius and the muon radius.[5]

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References

[1] S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity* (John Wiley & Sons, Inc. New York, 1972), pp. 40-41.

[2] http://www.electronformfactor.com/Mott-Rutherford Scattering and Beyond

[3] J.D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill, New York, 1964), pp. 108-111.

[4] J.D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill, New York, 1964), p 118.

[5] http://www.electronformfactor.com/Møller Scattering