# MØLLER SCATTERING

ABSTRACT

The electron charge is considered to be distributed or extended in space. The differential of the electron charge is set equal to a function of electron charge coordinates multiplied by a four-dimensional differential volume element. The four-dimensional integral of this function is required to equal the electron charge in all Lorentz frames. The probability amplitude for direct extended electron-electron scattering is calculated. The result is that the probability amplitude of the extended electron theory is a product of the probability amplitude of the point electron theory multiplied by two electron form factors. Due to the identity of the electrons, a second so-called "exchange" probability amplitude must be calculated. As before, this probability amplitude in the extended electron theory is a product of the probability amplitude of the point electron theory multiplied by two electron form factors. Due to Fermi statistics, the $S$-matrix is the difference of the two probability amplitudes .

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# I. INTRODUCTION

In the rest frame of an electron charge distribution, let $x_r'^{\mu} = (x_r'^0, x_r'^1, x_r'^2, x_r'^3)$ denote a spacetime charge point, and let $x_r^{\mu} = (x_r^0, x_r^1, x_r^2, x_r^3)$ denote the center of the charge distribution. Sometimes the superscript on the four-vector (not the components) will be omitted, and we will write $x_r' = (x_r'^0, x_r'^1, x_r'^2, x_r'^3)$ and $x_r = (x_r^0, x_r^1, x_r^2, x_r^3)$. Introduce $\tilde{x}_r = x_r' - x_r$ or equivalently $\tilde{x}_r^{\mu} = x_r'^{\mu} - x_r^{\mu}$. In a frame of reference in which the charge distribution moves with a speed $\beta$ in the $+x^3$ direction, let $x_m' = (x_m'^0, x_m'^1, x_m'^2, x_m'^3)$ denote a spacetime charge point, and let $x_m = (x_m^0, x_m^1, x_m^2, x_m^3)$ denote the center of the charge distribution. Introduce $\tilde{x}_m = x_m' - x_m$. A Lorentz transformation yields $\tilde{x}_r^1 = \tilde{x}_m^1$, $\tilde{x}_r^2 = \tilde{x}_m^2$, $\tilde{x}_r^3 = \gamma(\tilde{x}_m^3 - \beta\tilde{x}_m^0)$, and $\tilde{x}_r^0 = \gamma(\tilde{x}_m^0 - \beta\tilde{x}_m^3)$ where $\gamma = 1/\sqrt{1-\beta^2}$. Denote this Lorentz transformation by $\tilde{x}_r^{\mu} = L(\tilde{x}_m^{\mu})$.

In this paper, the word "invariant" shall mean a quantity which is unchanged by a Lorentz transformation. The scalar product of two four-vectors is invariant. An example is $x_r \cdot x_r = x_m \cdot x_m$. To establish this equality, use the Lorentz transformation to write $x_r$ in terms of $x_m$ on the left hand side.

In the rest frame, the electron charge $e$ is equal to $\int \rho_r(\tilde{x}_r)\delta(\tilde{x}_r^0)d^4\tilde{x}_r$ where $\rho_r(\tilde{x}_r)$ is the charge density in the rest frame and $\delta$ denotes the delta function.[1] In the m frame, the electric charge $e$ is equal to $\int \rho_r(L(\tilde{x}_m))\delta[\gamma(\tilde{x}^0_m - \beta\tilde{x}^3_m)]d^4\tilde{x}_m$ .[2] So an element of charge $de_m$ in the m frame is given by

$$de_m = \rho_r(L\tilde{x}_m))\delta[\gamma(\tilde{x}^0_m - \beta\tilde{x}^3_m)]d^4\tilde{x}_m. \quad (1)$$

The next section is a review of point electron-electron scattering. As pointed out in the Abstract, there will be a probability amplitude for direct scattering, and there will be a probability amplitude for exchange scattering. The $S$-matrix is the difference between the two probability amplitudes. The third section treats extended electron-electron scattering. The result is that each probability amplitude is the probability amplitude for point electron-electron scattering multiplied by two electron form factors. The $S$-matrix of the extended electron theory is again the the difference between the probability amplitudes for direct and exchange scattering. A short discussion follows where an estimate is made of the maximum electron radius and the maximum muon radius.

## II. ELECTRON-ELECTRON SCATTERING

The calculation for electron-electron scattering will follow the calculation of electron-proton scattering in Bjorken and Drell where the proton is treated as a point particle of spin 1/2.[3] The $S$-matrix for electron-proton scattering is

$$S_{fi} = -i \int d^4x\, \bar{\phi}_f(x)(e\gamma^\mu)A_\mu(x)\phi_i(x). \tag{2}$$

With a slight modification, this equation can be used to study electron-electron scattering since both electrons are treated here as a point particles of spin 1/2. The initial electron wave function, which has been normalized to unity in a box of volume V, is approximated by the plane wave

$$\phi_i(x) = \sqrt{\frac{m}{E_i V}}\, u_i \exp\left(-ip_i \cdot x\right) \tag{3}$$

where $\hbar$ and $c$ have been set equal to 1, $m$ is the electron rest mass, $u_i$ is a four-component spinor, which depends on the initial spin and on $p_i^\mu = (p_i^0 = E_i, p_i^1, p_i^2, p_i^3)$, the initial four-momentum, and $\gamma^\mu$ where $\mu = 0, 1, 2, 3$ are the four Dirac matrices. The final electron wave function is

$$\phi_f(x) = \sqrt{\frac{m}{E_f V}}\, u_f \exp\left(-ip_f \cdot x\right), \tag{4}$$

where $p_f$ is the final electron four-momentum, $E_f$ is the final electron energy, $u_f$ is the final electron spinor, and $\bar{\phi}_f = \phi_f^\dagger \gamma^0$. The vector potential is given by

$$A_\mu(x) = \int d^4y\, D_F(x-y)\, J_\mu(y) \tag{5}$$

where $D_F(x-y)$ is the photon propagator, and $J_\mu(y)$ is now the second electron current. The photon propagator is

$$D_F(x-y) = \int \frac{d^4q}{(2\pi)^4} \exp\left[-iq\cdot(x-y)\right] \frac{-1}{q^2+i\epsilon} \tag{6}$$

where $q$ is the four-momentum of the photon. The second electron current is identified as $\phi_F(y) e\gamma_\mu\, \phi_F(y)$ where $\phi_F(y)$ is the final elctron wave function and $\phi_I(y)$ is the initial electron wave function.

The initial second electron wave function is approximated by the plane wave

$$\phi_I(y) = \sqrt{\frac{m}{E_I V}}\, u_I \exp(-ip_I \cdot y) \tag{7}$$

where $u_I$ is a four component spinor, which depends on the initial electron spin and on $p_I^\mu = (p_I^0 = E_I, p_I^1, p_I^2, p_I^3)$, the initial elctron four-momentum. The final second electron wave function is

$$\phi_F(y) = \sqrt{\frac{m}{E_F V}}\, u_F \exp(-ip_F \cdot y) \tag{8}$$

where $p_F$ is the final electron momentum four-vector, $E_F$ is the final electron energy, and $u_F$ is the final four-component spinor of the electron. Thus, the probability amplitude for direct electron-electron scattering is

$$S_{fi1} = -i \int d^4x\, d^4y\, \bar{\phi}_f(x)(e\gamma^\mu)\phi_i(x)D_F(x-y)\bar{\phi}_F(y)(e\gamma_\mu)\phi_I(y). \tag{9}$$

The subscript 1 has been added to $S_{fi}$ since there are two probability amplitudes making up the $S$-matrix. In order to keep track of the imaginary i's in more complicated situations, the following convention will be adopted: each electron charge $e$ is replaced by $-ie$; and also $D_F(x-y)$ is replaced by $iD_F(x-y)$.[4] Thus

$$S_{fi1} = \int d^4x\, d^4y\, \bar{\phi}_f(x)(-ie\gamma^\mu)\phi_i(x)iD_F(x-y)\bar{\phi}_F(y)(-ie\gamma_\mu)\phi_I(y). \tag{10}$$

Substituting Eqs. (3), (4), (6), (7), and (8) into Eq. (9) or (10) yields

$$S_{fi1} = \frac{+ie^2m^2(\bar{u}_f\gamma^\mu u_i)(\bar{u}_F\gamma_\mu u_I)}{(2\pi)^4V^2\sqrt{E_iE_fE_IE_F}} \int \frac{d^4x\,d^4y\,d^4q\,\exp[i(p_f - p_i - q)\cdot x]\exp[i(p_F - p_I + q)\cdot y]}{q^2 + i\epsilon}. \quad (11)$$

Perform the following integrations:

$$\int \exp(i(p_f - p_i - q))d^4x = (2\pi)^4\delta^4(p_f - p_i - q); \quad (12)$$

$$\int \exp(i(p_F - p_I + q))d^4x = (2\pi)^4\delta^4(p_F - p_I + q); \quad (13)$$

$$\int \delta^4(p_f - p_i - q)\delta^4(p_F - p_I + q)\frac{d^4q}{q^2 + i\epsilon} = \frac{\delta^4(p_F + p_f - p_I - p_i)}{(p_f - p_i)^2}; \quad (14)$$

and find

$$S_{fi1} = \frac{+ie^2m^2}{V^2\sqrt{E_iE_fE_IE_F}}(2\pi)^4\delta^4(p_f + p_F - p_i - p_I)\frac{(\bar{u}_f\gamma^\mu u_i)(\bar{u}_F\gamma_\mu u_I)}{(p_f - p_i)^2}. \quad (15)$$

By the delta function in Eq. (12), $q = p_f - p_i$.

The $S$-matrix, $S_{fi}$, contains a second probability amplitude, $S_{fi2}$, which has the final wave functions exchanged due to the identity of the electrons.[5] Thus

$$S_{fi2} = \int d^4x\, d^4y\, \bar{\phi}_F(x)(-ie\gamma^\mu)\phi_i(x)iD_F(x-y)\bar{\phi}_f(y)(-ie\gamma_\mu)\phi_I(y). \tag{16}$$

Substituting Eqs. (3), (4), (6), (7), and (8) into Eq. (16) yields

$$S_{fi2} = \frac{+ie^2m^2(\bar{u}_F\gamma^\mu u_i)(\bar{u}_f\gamma_\mu u_I)}{(2\pi)^4V^2\sqrt{E_iE_fE_IE_F}} \int \frac{d^4x\, d^4y\, d^4q\, \exp[i(p_F - p_i - q)\cdot x]\exp[i(p_f - p_I + q)\cdot y]}{q^2 + i\epsilon}. \tag{17}$$

The resulting delta function in the above equation yields $q = p_F - p_i$. Perform the integrations and get

$$S_{fi2} = \frac{+ie^2m^2}{V^2\sqrt{E_iE_fE_IE_F}}(2\pi)^4\delta^4(p_f + p_F - p_i - p_I)\frac{(\bar{u}_F\gamma^\mu u_i)(\bar{u}_f\gamma_\mu u_I)}{(p_F - p_i)^2}. \tag{18}$$

Due to Fermi statistics, the $S$-matrix is $S_{fi} = S_{fi1} - S_{fi2}$.

## III. EXTENDED ELECTRON-ELECTRON SCATTERING

Suppose that the first electron initially moves with a speed $\beta$ in the $+x^3$ direction. This is the m frame. Recall that $x'_m$ is a spacetime electron charge point, $x_m$ is the center of the electron and also the argument of the wave function. Define $\tilde{x}_m = x'_m - x_m$. In the rest frame of the second electron, let $y'_r$ be a spacetime charge point, and let $y_r$ be the center of the electron charge. Introduce $\tilde{y}_r = y'_r - y_r$. Take the second electron to be moving with a speed $\beta$ in the $-y^3$ direction in the m frame. The m frame is now the center of mass frame. Let $y'_m$ be a spacetime charge point of the second electron. Let $y_m$ be the center of the second electron and the argument of the wave function. Introduce $\tilde{y}_m = y'_m - y_m$. A Lorentz transformation yields $\tilde{y}^0_r = \gamma(\tilde{y}^0_m + \beta\tilde{y}^3_m)$, $\tilde{y}^1_r = \tilde{y}^1_m$, $\tilde{y}^2_r = \tilde{y}^2_m$, and $\tilde{y}^3_r = \gamma(\tilde{y}^3_m + \beta\tilde{y}^0_m)$ where $\gamma = 1/\sqrt{1-\beta^2}$. Denote this Lorentz transformation by $\tilde{y}_r = L(\tilde{y}_m)$.

The interaction takes place at charge points, so replace $D_F(x-y)$ by $D_F(x'_m - y'_m)$ in Eq. (9). Let the photon four-momentum vector now be labeled $q_m$. Also, replace the first electron charge by the four-dimensional integral of $de_{m1} = \rho_r(L(\tilde{x}_m))\delta(\gamma(\tilde{x}^0_m - \beta\tilde{x}^3_m))d^4\tilde{x}_m$, and replace the second electron charge by the four-dimensional integral of $de_{m2} = \rho_r(L(\tilde{y}_m))\delta(\gamma(\tilde{y}^0_m + \beta\tilde{y}^3_m))d^4\tilde{y}_m$. So now the first probability amplitude $S_{FI1}$ is

$$S_{FI1} = \int d^4x_m\, d^4y_m\, \bar{\phi}_f(x_m)(-i\gamma^\mu)\phi_i(x_m)de_{m1}$$
$$iD_F(x'_m - y'_m)\bar{\phi}_F(y_m)(-i\gamma_\mu)\phi_I(y_m)de_{m2}. \quad (19)$$

Use $D_F(x'_m - y'_m) = D_F(x_m - y_m)\exp(-i\tilde{x}_m \cdot q_m)\exp(+i\tilde{y}_m \cdot q_m)$ to show

$$
\begin{aligned}
S_{FI1} = \int d^4x_m\, d^4y_m\, \bar{\phi}_f(x_m)(-ie\gamma^\mu)\phi_i(x_m) iD_F(x_m - y_m)\bar{\phi}_F(y_m)(-ie\gamma_\mu)\phi_I(y_m) \\
\int \exp(-i\tilde{x}_m \cdot q_m)\frac{\rho_r(L(\tilde{x}_m))}{e}\delta(\gamma(\tilde{x}^0_m - \beta\tilde{x}^3_m))d^4\tilde{x}_m \\
\int \exp(+i\tilde{y} \cdot q_m)\frac{\rho_r(L(\tilde{y}_m))}{e}\delta(\gamma(\tilde{y}^0_m + \beta\tilde{y}^3_m))d^4\tilde{y}_m\,. \quad (20)
\end{aligned}
$$

By the delta function which results when integrating the first integral above, $q_m = p_f - p_i$, and by Eqs. (9) and (15), the first integral is $S_{fi1}$, so

$$
S_{FI1} = S_{fi1}F_1(q)F_2(q) \tag{21}
$$

where

$$
F_1(q) = \int \exp(-i\tilde{x}_m \cdot q_m)\frac{\rho_r(L(\tilde{x}_m))}{e}\delta(\gamma(\tilde{x}^0_m - \beta\tilde{x}^3_m))d^4\tilde{x}_m \tag{22}
$$

is the first electron form factor and

$$
F_2(q) = \int \exp(+i\tilde{y}_m \cdot q_m)\frac{\rho_r(L(\tilde{y}_m))}{e}\delta(\gamma(\tilde{y}^0_m + \beta\tilde{y}^3_m))d^4\tilde{y}_m \tag{23}
$$

is the second electron form factor. Due to invariance of the form factor,[2]

$$F_1(q) = \int \exp(-i\tilde{x}_r \cdot q_r) \frac{\rho_r(\tilde{x}_r)}{e} \delta(\tilde{x}_r^0) d^4\tilde{x}_r \tag{24}$$

and

$$F_2(q) = \int \exp(+i\tilde{y}_r \cdot q_r) \frac{\rho_r(\tilde{y}_r))}{e} \delta(\tilde{y}_r^0) d^4\tilde{y}_r\,, \tag{25}$$

where $q_r$ is related to $q_m$ by a Lorentz transformation.

To get a rough idea of how size and structure affect scattering, pick the electron charge to be uniformly distributed on a spherical shell of radius $a$ in the rest frame. Then the charge densities will be given by $\rho_r(\tilde{x}_r) = e\delta(|\tilde{\mathbf{x}}_r| - a)/4\pi\, a^2$ and $\rho_r(\tilde{y}_r) = e\delta(|\tilde{\mathbf{y}}_r| - a)/4\pi\, a^2$. The results are $F_1(q) = F_2(q) = j_0(|\mathbf{q}_r|a)$ where $j_0$ is the spherical Bessel function of order zero, and

$$|\mathbf{q}_r|^2 = (q_m^1)^2 + (q_m^2)^2 + \gamma^2(q_m^3 - \beta q_m^0)^2 = \frac{(p_f \cdot p_i)^2 - m^4}{m^2}.^2 \tag{26}$$

For relativistic speeds, the electron mass can be neglected, so $E_i = |\mathbf{p}_i|$ and $E_f = |\mathbf{p}_f|$. In the center of mass frame, $|\mathbf{p}_f| = |\mathbf{p}_i|$, so $E_f = E_i$. Then

$$|\mathbf{q}_r|^2 \approx \frac{[E_f E_i(1 - \cos(\theta))]^2}{m^2} = \frac{4E_f^2 E_i^2 \sin^4(\theta/2)}{m^2} \tag{27}$$

Put back $\hbar$ and $c$ in Eq. (27), use $|\mathbf{p_i}| \approx \gamma mc$ and find

$$|\mathbf{q}_r| \approx \frac{2(mc)\sin^2(\theta/2)}{\hbar(1-\beta^2)}. \tag{28}$$

The second probability amplitude, $S_{FI2}$, is given by

$$\begin{aligned} S_{FI2} = \int d^4x_m\, d^4y_m\, \bar{\phi}_F(x_m)(-i\gamma^\mu)\phi_i(x_m)de_{m1} \\ iD_F(x'_m - y'_m)\bar{\phi}_f(y_m)(-i\gamma_\mu)\phi_I(y_m)de_{m2} \end{aligned} \tag{29}$$

where the photon momentum four-vector is now written $q_{me}$. This equation can be written as

$$\begin{aligned} S_{FI2} = \int d^4x_m\, d^4y_m\, \bar{\phi}_F(x_m)(-ie\gamma^\mu)\phi_i(x_m)iD_F(x_m - y_m)\bar{\phi}_f(y_m)(-ie\gamma_\mu)\phi_I(y_m) \\ \int \exp(-i\tilde{x}_m \cdot q_{me})\frac{\rho_r(L(\tilde{x}_m))}{e}\delta(\gamma(\tilde{x}^0_m - \beta\tilde{x}^3_m))d^4\tilde{x}_m \\ \int \exp(+i\tilde{y} \cdot q_{me})\frac{\rho_r(L(\tilde{y}_m))}{e}\delta(\gamma(\tilde{y}^0_m + \beta\tilde{y}^3_m))d^4\tilde{y}_m\,. \end{aligned} \tag{30}$$

By the delta function which results when integrating the first integral above, $q_{me} = p_F - p_i$. Integration leads to

$$S_{FI2} = S_{fi2}F_{1e}(q)F_{2e}(q)\,. \tag{31}$$

where

$$F_{1e}(q) = \int \exp(-i\tilde{x}_m \cdot q_{me}) \frac{\rho_r(L(\tilde{x}_m))}{e} \delta(\gamma(\tilde{x}_m^0 - \beta \tilde{x}_m^3)) d^4 \tilde{x}_m \tag{32}$$

and

$$F_{2e}(q) = \int \exp(+i\tilde{y}_m \cdot q_{me}) \frac{\rho_r(L(\tilde{y}_m))}{e} \delta(\gamma(\tilde{y}_m^0 + \beta \tilde{y}_m^3)) d^4 \tilde{y}_m \tag{33}$$

Again take electron charge to be distributed uniformly on a spherical shell of radius $a$, $F_{1e}(q) = F_{2e}(q) = j_0(|\mathbf{q}_{re}|a)$. Repeating a previous calculation,

$$|\mathbf{q}_{re}|^2 = (q_{me}^1)^2 + (q_{me}^2)^2 + \gamma^2(q_{me}^3 - \beta q_{me}^0)^2 = \frac{(p_F \cdot p_i)^2 - m^4}{m^2}.^2 \tag{34}$$

Note that $|\mathbf{p}_F| = |\mathbf{p}_i| = E_F = E_i$ for relativistic speeds in the center of mass frame. The angle of scattering in the exchange process is $\theta_e = \theta + \pi$. Then $\cos\theta_e = -\cos\,\theta$, and

$$|\mathbf{q}_{re}|^2 \approx \frac{[E_f E_i(1 - \cos{(\theta_e)})]^2}{m^2} = \frac{4E_f^2 E_i^2 \cos^4{(\theta/2)}}{m^2} \tag{35}$$

Due to Fermi statistics, $S_{FI2}$ must be subtracted from $S_{FI1}$ to get the $S$-matrix, $S_{FI}$. Therefore $S_{FI} = S_{FI1} - S_{FI2}$.

## IV. DISCUSSION

The $S$-matrix for electron-electron scattering is the difference of two probability amplitudes. In the extended electron theory, each of the point electron probability amplitudes is multiplied by two electron form factors. A specific electron charge density was chosen just to get an idea of how electron size and structure affect electron scattering. The actual charge density should be calculated from the experimentally determined form factor.

The calculation below is a rough estimate of the maximum electron radius. The $S$-matrix for extended electron-electron scattering depends on powers of $j_0(|\mathbf{q}_r|a)$ and on powers of $j_0(|\mathbf{q}_{er}|a)$. Neglecting the $\theta$ dependence, $|\mathbf{q}_r| = |\mathbf{q}_{re}|$. For small argument, each form factor is approximately $1 - (|\mathbf{q}_r|a)^2/6$. The point electron theory of quantum electrodynamics is a very successful theory. Size effects appear to be absent in electron-electron scattering. This suggests that $(|\mathbf{q}_r|a)^2/6$ is too small to be detected experimentally. Assume the undetectable value of $|\mathbf{q}_r|a$ to be $< .1$ . For 40 GeV electrons, $1-\beta^2 \approx 1 \cdot 10^{-10}$, and $\beta \approx 1$. Then

$$|\mathbf{q}_r| \approx \frac{mc}{\hbar(1-\beta^2)} \approx 1 \cdot 10^{23}\,(\text{meters})^{-1} \tag{36}$$

If the electron radius $a \approx 1 \cdot 10^{-24}$ meters, then $|\mathbf{q}_r|a \approx .1$. This suggests that the electron radius is less than $1 \cdot 10^{-24}$ meters.

Repeat the calculation for the muon using Eq. (36). The muon mass is about 200 times the electron mass. Take the muon speed to be .997 times the speed of light. Then, $|\mathbf{q}_r| \approx 1 \cdot 10^{+17}(\text{meters})^{-1}$, so $a_\mu < 1 \cdot 10^{-18}$ meters.

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