POSITRON SCATTERING BY A COULOMB POTENTIAL

Abstract

The purpose of this short paper is to show how positrons are treated in quantum electrodynamics, and to study how positron size affects scattering. The positron charge is considered to be distributed or extended in space. The differential of the positron charge is set equal to a function of positron charge coordinates multiplied by a four-dimensional differential volume element. The four-dimensional integral of this function is required to equal the positron charge in all Lorentz frames. The S-matrix for the scattering of such a positron by a Coulomb potential is calculated. The result is that the S-matrix of the extended positron theory is a product of the S-matrix of the point positron theory multiplied by a positron form factor.

I. INTRODUCTION

In the rest frame of a positron charge distribution, let $x_r'^{\nu} = (x_r'^0, x_r'^1, x_r'^2, x_r'^3)$ denote a spacetime charge point, and let $x_r^{\nu} = (x_r^0, x_r^1, x_r^2, x_r^3)$ denote the center of the charge distribution. Sometimes the superscript will be omitted, and we will write $x_r' = (x_r'^0, x_r'^1, x_r'^2, x_r'^3)$ and $x_r = (x_r^0, x_r^1, x_r^2, x_r^3)$.

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Introduce $\tilde{x}_r = x'_r - x_r$ or equivalently $\tilde{x}^\nu_r = x'^\nu_r - x^\nu_r$. In a frame of reference in which the positron charge distribution moves with a speed β in the $+x^3$ direction, let $x'_m = (x'^0_m, x'^1_m, x'^2_m, x'^3_m)$ denote a spacetime charge point, and let $x_m = (x^0_m, x^1_m, x^2_m, x^3_m)$ denote the center of the charge distribution. Introduce $\tilde{x}_m = x'_m - x_m$. A Lorentz transformation yields $\tilde{x}^1_r = \tilde{x}^1_m$, $\tilde{x}^2_r = \tilde{x}^2_m$, $\tilde{x}^3_r = \gamma(\tilde{x}^3_m - \beta\tilde{x}^0_m)$, and $\tilde{x}^0_r = \gamma(\tilde{x}^0_m - \beta\tilde{x}^3_m)$ where $\gamma = 1/\sqrt{1-\beta^2}$. Denote this Lorentz transformation by $\tilde{x}_r = L(\tilde{x}_m)$.

In the rest frame, the positron charge $-e > 0$ is equal to $\int \rho_r(\tilde{x}_r)\delta(\tilde{x}^0_r)d^4\tilde{x}_r$ where $\rho_r(\tilde{x}_r)$ is the charge density in the rest frame and δ denotes the delta function.[1] In the m frame, the positron charge $-e$ is equal to $\int \rho_r(L(\tilde{x}_m))\delta[\gamma(\tilde{x}^0_m - \beta\tilde{x}^3_m)]d^4\tilde{x}_m$.[2] So an element of charge de^+_m in the m frame is given by

$$de^+_m = \rho_r(L(\tilde{x}_m))\delta[\gamma(\tilde{x}^0_m - \beta\tilde{x}^3_m)]d^4\tilde{x}_m. \tag{1}$$

The next section reviews the calculation of the S-matrix for point positron scattering by a Coulomb potential. In the third section, the S-matrix for the extended positron in a Coulomb potential is calculated. The S-matrix of the extended positron theory is found to be the product of the S-matrix of the point positron theory multiplied by a positron form factor.

II. COULOMB SCATTERING OF POSITRONS

The calculation of the S-matrix for the scattering of a positron from a fixed Coulomb potential will follow Bjorken and Drell.[3] For a point positron, the S-matrix element is

$$S_{fi} = +i \int d^4x \, \bar{\phi}_F(x)(e\gamma^0)A_0(\mathbf{x})\phi_I(x). \tag{2}$$

Let $p_I = (E_I, p_I^1, p_I^2, p_I^3)$ denote the initial positron four-momentum vector, and let $p_F = (E_F, p_F^1, p_F^2, p_F^3)$ denote the final positron four-momentum vector. It is common to treat the positive energy positron, which propagates forward in time, as a negative energy electron, which propagates backward in time.[4] Thus the negative energy electron propagates from the future into the past. To accommodate this, the initial four-momentum of the electron is identified as $-p_F = (-E_F, -\mathbf{p}_F)$, and the final four-momentum of the electron is identified as $-p_I = (-E_I, -\mathbf{p}_I)$. For the electron moving backward in time, notice that the S-matrix element has a $+i$ in front of the integral.[5] The initial electron plane wave function, which has been normalized to unity in a box of volume V, now is

$$\phi_I(x) = \sqrt{\frac{m}{E_F V}} \, v_F \exp\left(+ip_F \cdot x\right), \tag{3}$$

and the final electron plane wave function is

$$\phi_F(x) = \sqrt{\frac{m}{E_I V}}\, v_I \exp\left(+ip_I \cdot x\right) \tag{4}$$

Here $\hbar$ and c are set equal to 1, m is the positron mass, v_F is a one by four column matrix, which depends on the final positron spin and on the final positron four-momentum, and v_I is a one by four column matrix, which depends on the initial positron spin and on the initial four-momentum. The four by four Dirac matrix $\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Each entry in γ^0 is actually a two by two matrix.Also $\bar{\phi}_F = \phi_F^\dagger \gamma^0$.

For a point charge $-Ze > 0$, the Coulomb potential is $A_0(\mathbf{x}) = -Ze/4\pi|\mathbf{x}|$. Here $\mathbf{x} = (x^1, x^2, x^3)$, so $|\mathbf{x}| = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}$. Then

$$S_{fi} = -\frac{iZe^2 m}{V\sqrt{E_I E_F}}\, \bar{v}_I \gamma^0 v_F \int \frac{\exp\left[i(p_F - p_I)\cdot x\right] d^4x}{4\pi|\mathbf{x}|}. \tag{5}$$

Let $q = p_F - p_I$. Write $\exp\left(iq \cdot x\right) = \exp\left[i(q^0 x^0 - \mathbf{q}\cdot\mathbf{x})\right]$. Perform the following integrations:

$$\int \exp\left(iq^0 x^0\right) dx^0 = 2\pi\delta(q^0) = 2\pi\delta(E_F - E_I); \tag{6}$$

$$\int \frac{\exp\left(-i\mathbf{q}\cdot\mathbf{x}\right)}{|\mathbf{x}|} d^3x = \frac{4\pi}{|\mathbf{q}|^2} = \frac{4\pi}{|\mathbf{p}_F - \mathbf{p}_I|^2}; \tag{7}$$

and find

$$S_{fi} = -\frac{iZe^2m}{V\sqrt{E_I E_F}}\frac{\bar{v}_I\gamma^0 v_F}{|\mathbf{q}|^2}2\pi\delta(E_F - E_I). \tag{8}$$

The integral performed in Eq. (7) is the Fourier transform of a generalized function.[6]

III. COULOMB SCATTERING OF EXTENDED POSITRONS

In the previous section, x was the argument of the wave function and the negative energy electron charge spacetime point in an arbitrary Lorentz frame. In this section, take the positron to be moving in the $+x^3$ direction with a speed β. This frame was previously defined to be the m frame. So now x_m is the argument of the wave function, and it is also the center of the positron charge distribution.

Eq. (2) will be modified to take into account the spatial distribution of the electron charge. The interaction takes place at the charge points x'_m, so $A_0(\mathbf{x})$ is replaced by

$$A_0(\mathbf{x}'_m) = \frac{-Ze}{4\pi|\mathbf{x}'_m|}\,. \tag{9}$$

The negative energy electron moving backward in time has a speed β in the $-x^3$ direction. A Lorentz transformation to the rest frame of the negative energy electron is denoted by $\tilde{x}_{re} = L^-(\tilde{x}_m)$. Let ρ_r

be the charge density of the electron in the rest frame. The negative energy electron charge is replaced by the four-dimensional integral of $\rho_r(L^-(\tilde{x}_m))\delta[\gamma(\tilde{x}^0_m + \beta\tilde{x}^3_m)]d^4x_m$. Making these substitutions in Equation (2), the S matrix for the extended positron is

$$S_{FI} = +i\int d^4x_m\,\bar{\phi}_F(x_m)\rho_r(L^-(\tilde{x}_m))\delta[\gamma(\tilde{x}^0_m+\beta\tilde{x}^3_m)]d^4\tilde{x}_m\,(\gamma^0)\frac{-Ze}{(4\pi|\mathbf{x}'_m|)}\phi_I(x_m). \tag{10}$$

The S-matrix becomes

$$S_{FI} = -\frac{iZem}{V\sqrt{E_IE_F}}\bar{v}_I\gamma^0v_F\int\frac{d^4x_m\exp\left[i(p_F-p_I)\cdot x_m\right]}{4\pi|\mathbf{x}'_m|}$$
$$\rho_r(L^-(\tilde{x}_m))\delta[\gamma(\tilde{x}^0_m+\beta\tilde{x}^3_m)]d^4\tilde{x}_m. \tag{11}$$

Change variables from x_m to $x'_m = x_m + \tilde{x}_m$. Since p_F and p_I are now measured in the m frame, set $q_m = p_F - p_I$.

$$S_{FI} = -\frac{iZem}{V\sqrt{E_iE_f}}\bar{v}_I\gamma^0v_F\int\frac{\exp\left(+iq_m\cdot x'_m\right)d^4x'_m}{4\pi|\mathbf{x}'_m|}$$
$$\int\exp\left(-iq_m\cdot\tilde{x}_m\right)\rho_r(L^-(\tilde{x}_m))\delta[\gamma(\tilde{x}^0_m+\beta\,\tilde{x}^3_m]d^4\tilde{x}_m. \tag{12}$$

By Eqns, (6) and (7)

$$S_{FI} = S_{fi}F(q) \tag{13}$$

where $F(q)$, the positron form factor, is given by

$$F(q) = \frac{1}{e}\int \exp\left(-iq_m \cdot \tilde{x}_m\right)\rho_r(L^{-}(\tilde{x}_m))\delta[\gamma_i(\tilde{x}^0_m + \beta_i\,\tilde{x}^3_m]d^4\tilde{x}_m. \quad (14)$$

The following result has been established in reference 2

$$F(q) = \frac{1}{e}\int \exp\left(-iq_r \cdot \tilde{x}_{re}\right)\rho_r(\tilde{x}_{re}))\delta(x^0_{re})d^4\tilde{x}_{re} \quad (15)$$

where $q_r = L(q) = (\gamma(q^0 - \beta q^3), q^1, q^2, \gamma(q^3 - \beta q^0))$ is the q four-vector in the rest frame.

To get a rough idea of how size affects scattering, assume the electron charge is uniformly distributed on a spherical shell of radius a in the rest frame. Then

$$\rho_r(\tilde{x}_r) = \frac{e}{4\pi a^2}\delta(\sqrt{(\tilde{x}^1_r)^2 + (\tilde{x}^2_r)^2 + (\tilde{x}^3_r)^2} - a), \quad (16)$$

and $F(q) = j_0(|\mathbf{q}_r|a)$ where j_0 is the spherical Bessel function of order zero, and

$$|\mathbf{q}_r|^2 = (p^1_F)^2 + (p^2_F)^2 + \gamma^2(p^3_F - p^3_I - \beta(p^0_F - p^0_I))^2 = \frac{(p_F \cdot p_I)^2 - m^4}{m^2} = \frac{4|\mathbf{p}^2_I|\sin^2(\theta/2)(1-\beta^2\cos^2(\theta/2))}{(1-\beta^2)}. \quad (17)$$

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References

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