ELECTRON-PROTON SCATTERING

ABSTRACT

The electron charge is considered to be distributed or extended in space. The differential of the electron charge is set equal to a function of electron charge coordinates multiplied by a four-dimensional differential volume element. The four-dimensional integral of this function is required to equal the electron charge in all Lorentz frames. The structure of the proton can be written in terms of two proton form factors. The S-matrix for the extended electron and structured proton is calculated. The result is that the S-matrix contains three form factors.

I. INTRODUCTION

In the rest frame of an electron charge distribution, let $x_r'^{\mu} = (x_r'^{0}, x_r'^{1}, x_r'^{2}, x_r'^{3})$ denote a spacetime charge point, and let $x_r^{\mu} = (x_r^0, x_r^1, x_r^2, x_r^3)$ denote the center of the charge distribution. Sometimes the superscript on the four-vector (not the components) will be omitted, and we will write

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$x'_r = (x'^0_r, x'^1_r, x'^2_r, x'^3_r)$ and $x_r = (x^0_r, x^1_r, x^2_r, x^3_r)$. Introduce $\tilde{x}_r = x'_r - x_r$ or equivalently $\tilde{x}^\mu_r = x'^\mu_r - x^\mu_r$. In a frame of reference in which the electron charge distribution moves with a speed β in the $+x^3$ direction, let $x'_m = (x'^0_m, x'^1_m, x'^2_m, x'^3_m)$ denote a spacetime charge point, and let $x_m = (x^0_m, x^1_m, x^2_m, x^3_m)$ denote the center of the charge distribution. Introduce $\tilde{x}_m = x'_m - x_m$. A Lorentz transformation yields $\tilde{x}^1_r = \tilde{x}^1_m$, $\tilde{x}^2_r = \tilde{x}^2_m$, $\tilde{x}^3_r = \gamma(\tilde{x}^3_m - \beta\tilde{x}^0_m)$, and $\tilde{x}^0_r = \gamma(\tilde{x}^0_m - \beta\tilde{x}^3_m)$ where $\gamma = 1/\sqrt{1-\beta^2}$. Denote this Lorentz transformation by $\tilde{x}_r = L(\tilde{x}_m)$.

In the rest frame, the electron charge e is equal to $\int \rho_r(\tilde{x}_r)\delta(\tilde{x}^0_r)d^4\tilde{x}_r$ where $\rho_r(\tilde{x}_r)$ is the charge density in the rest frame and δ denotes the delta function.[1] In the m frame, the electric charge e is equal to $\int \rho_r(L(\tilde{x}_m))\delta[\gamma(\tilde{x}^0_m - \beta\tilde{x}^3_m)]d^4\tilde{x}_m$.[2] So an element of charge de_m in the m frame is given by

$$de_m = \rho_r(L(\tilde{x}_m))\delta[\gamma(\tilde{x}^0_m - \beta\tilde{x}^3_m)]d^4\tilde{x}_m. \tag{1}$$

The next section is a review of point electron scattering by a point proton. The third section will study scattering of a point electron by a structured proton. In the fourth section, the S-matrix of an extended electron and a structured proton is calculated. The result is that the S-matrix is the S-matrix of point electron scattering by the structured proton multiplied by an electron form factor. A short discussion follows.

II. POINT ELECTRON-POINT PROTON SCATTERING

The calculation for electron-proton scattering will follow the calculation of electron-proton scattering in Bjorken and Drell.[3] When the point electron at spacetime point x exchanges a photon with the point proton at spacetime point y, the S matrix element is approximated by

$$S_{fi} = \int d^4x \, d^4y \, \bar{\phi}_f(x)(-ie\gamma_\mu)) \, \phi_i(x) i D_F(x-y) \phi_F(y)(-ie\gamma^\mu)\phi_I(y). \tag{2}$$

The initial exact electron wave function is approximated by the plane wave solution to the Dirac equation. The plane wave solution, which is normalized to unity in a box of volume V, is

$$\phi_i(x) = \sqrt{\frac{m}{E_i V}} \, u_i \exp\left(-ip_i \cdot x\right) \tag{3}$$

where $\hbar$ and c have been set equal to 1, m is the electron rest mass, u_i is a four-component spinor, which depends on the initial spin and on $p_i = (p_i^0 = E_i, p_i^1, p_i^2, p_i^3)$, the initial four-momentum, and γ^μ are the four Dirac matrices, which are labelled by $\mu = 0, 1, 2, 3$. The final electron wave function is

$$\phi_f(x) = \sqrt{\frac{m}{E_f V}}\, u_f \exp\left(-ip_f \cdot x\right), \tag{4}$$

where p_f is the final electron four-momentum, E_f is the final electron energy, u_f is the final electron spinor, and $\bar{\phi}_f = \phi_f^\dagger \gamma^0$. The photon propagator is

$$D_F(x-y) = \int \frac{d^4 q}{(2\pi)^4} \exp\left[-iq \cdot (x-y)\right] \frac{-1}{q^2 + i\epsilon} \tag{5}$$

where q is the four-momentum of the photon, and the spacetime point y is the the argument of the proton wave function. The initial approximate proton wave function is

$$\phi_I(y) = \sqrt{\frac{M}{E_I V}}\, u_I \exp(-ip_I \cdot y) \tag{6}$$

where M is the proton mass, u_I is a four component spinor, which depends on the initial proton spin and on $p_I = (p_I^0 = E_I, p_I^1, p_I^2, p_I^3)$, the initial proton four-momentum. The final proton wave function is

$$\phi_F(y) = \sqrt{\frac{M}{E_F V}}\, u_F \exp(-ip_F \cdot y) \tag{7}$$

where p_F is the final proton momentum four-vector,E_F is the final proton energy, and u_F is the final four-component spinor of the proton. Substituting Eqs. (3), (4), (5), (6), and (7) into Eq. (2) yields

$$S_{fi} = \frac{+ie^2 mM(\bar{u}_f \gamma^\mu u_i)(\bar{u}_F \gamma_\mu u_I)}{(2\pi)^4 V^2 \sqrt{E_i E_f E_I E_F}} \int \frac{d^4x\, d^4y\, d^4q\, \exp[i(p_f - p_i - q)\cdot x]\exp[i(p_F - p_I + q)\cdot y]}{q^2 + i\epsilon}. \quad (8)$$

Perform the following integrations:

$$\int \exp(i(p_f - p_i - q)\cdot x)d^4x = (2\pi)^4\delta^4(p_f - p_i - q); \quad (9)$$

$$\int \exp(i(p_F - p_I + q)\cdot y)d^4y = (2\pi)^4\delta^4(p_F - p_I + q); \quad (10)$$

$$\int \delta^4(p_f - p_i - q)\delta^4(p_F - p_I + q)\frac{d^4q}{q^2 + i\epsilon} = \frac{\delta^4(p_F + p_f - p_I - p_i)}{(p_f - p_i)^2}; \quad (11)$$

and find

$$S_{fi} = \frac{+ie^2 mM}{V^2\sqrt{E_i E_f E_I E_F}}(2\pi)^4\delta^4(p_f + p_F - p_i - p_I)\frac{(\bar{u}_f \gamma^\mu u_i)(\bar{u}_F \gamma_\mu u_I)}{(p_f - p_i)^2}. \quad (12)$$

III. POINT ELECTRON-STRUCTURED PROTON SCATTERING

The S-matrix for point electron-structured proton scattering is [4]

$$S_{fip} = \int d^4x\, d^4y\, \bar{\phi}_f(x)(-ie\gamma_\mu))\, \phi_i(x) iD_F(x-y)$$
$$\phi_F(y)(+ie)\Big(\gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2M}F_2(q^2)\Big)\phi_I(y). \quad (13)$$

where $q = p_f - p_i$, $\sigma^{\mu\nu} = i[\gamma^\mu, \gamma^\nu]/2$, and F_1 and F_2 are proton form factors, which are to be determined from experiment. Substituting Eqs. (3), (4), (5), (6), and (7) into Eq. (15) yields

$$S_{fip} = \frac{+ie^2mM}{V^2\sqrt{E_iE_fE_IE_F}}\frac{(2\pi)^4\delta^4(p_f + p_F - p_i - p_I)}{(p_f - p_i)^2}$$
$$(\bar{u}_f\gamma_\mu u_i)\Big(\bar{u}_F\big(\gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2M}F_2(q^2)\big)u_I\Big). \quad (14)$$

IV. EXTENDED ELECTRON-STRUCTURED PROTON SCATTERING

In the previous section, x was the argument of the electron wave function and the electron charge spacetime point in an arbitrary Lorentz frame. Take the electron to be initally moving with a speed β in the $+x^3$ direction. This previously was called the m frame. So now x_m is the argument of the wave function and also the center of the electron charge distribution.

Suppose that the proton is initially at rest in the m frame. Let $y_r = (y_r^0, y_r^1, y_r^2, y_r^3)$ denote the argument of the proton wave function in the m frame The subscript r is attached to y to emphasize that the proton is at rest in the m frame.

Eq. (15) will be modified to take into account the spatial charge distribution of the electron. The interaction takes place at charge points, so replace $D_F(x-y)$ by $D_F(x'_m - y_r)$. In addition, the electron charge is replaced by the four-dimensional integral of de_m where de_m is given by Eq. (1). Then

$$
\begin{aligned}
S_{FIp} = \int d^4x_m\, d^4y_r\, \bar{\phi}_f(x_m)(-ide_m\gamma_\mu))\, \phi_i(x_m) iD_F(x'_m - y_r) \\
\phi_F(y_r)(+ie)\Big(\gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2M}F_2(q^2)\Big)\phi_I(y_r). \quad (15)
\end{aligned}
$$

Use $D_F(x'_m - y_r) = D_F(x_m - y_r)\exp\left(-iq_m \cdot \tilde{x}_m\right)$, and $q_m = p_f - p_i$ as measured in the m frame. Then

$$
S_{FIp} = S_{fip}F(q) \quad (16)
$$

where the electron form factor $F(q)$ is given by

$$F(q) = \int \exp\left(-iq_m \cdot \tilde{x}_m\right) \frac{de_m}{e} \tag{17}$$

By invariance of the form factor

$$F(q) = \int \exp\left(-iq_r \cdot \tilde{x}_r\right) \frac{de_r}{e} = \int \exp\left(-iq_r \cdot \tilde{x}_r\right) \frac{\rho_r(\tilde{x}_r)}{e} \delta(\tilde{x}_r^0)\, d^4\tilde{x}_r \tag{18}$$

where q_r is related to q_m by a Lorentz transformation. In addition to the two proton form factors, there is an electron form factor to be determined by experiment.

V. DISCUSSION

For the extended electron, the result is that the S-matrix for elastic electron-proton scattering is multiplied by the additional electron form factor. As a result, the differential cross section will be multiplied by the absolute value of electron form factor squared.

Similarly for the extended electron, the deep inelastic scattering differential cross section will now be multiplied by the absolute value of the electron form factor squared.

This analysis can be applied to muon-proton scattering. Perhaps muon size and electron size will explain the shrinking proton problem.

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References

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