PAIR ANNIHILATION

Abstract

The electron charge is considered to be distributed or extended in space. The differential of the electron charge is set equal to a function of electron charge coordinates multiplied by a four-dimensional differential volume element. The four-dimensional integral of this function is required to equal the electron charge in all Lorentz frames. The positron is also considered to be distributed or extended in space, and is treated similarly. The S-matrix for annihilation of an extended electron-positron pair, which results in an extended muon pair, is calculated. The result is that the S-matrix for extended leptons is the S-matrix for point leptons multiplied by an electron form factor and multiplied by a muon form factor.

I. INTRODUCTION

In the rest frame of an electron charge distribution, let $x_r'^{\nu} = (x_r'^{0}, x_r'^{1}, x_r'^{2}, x_r'^{3})$ denote a spacetime charge point, and let $x_r^{\nu} = (x_r^0, x_r^1, x_r^2, x_r^3)$ denote the center of the charge distribution. Sometimes the superscript on the four-vector (not the components) will be omitted, and we will write

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$x'_r = (x'^0_r, x'^1_r, x'^2_r, x'^3_r)$ and $x_r = (x^0_r, x^1_r, x^2_r, x^3_r)$. Introduce $\tilde{x}_r = x'_r - x_r$ or equivalently $\tilde{x}^\nu_r = x'^\nu_r - x^\nu_r$. In a frame of reference in which the electron charge distribution moves with a speed β in the $+x^3$ direction, let $x'_m = (x'^0_m, x'^1_m, x'^2_m, x'^3_m)$ denote a spacetime charge point, and let $x_m = (x^0_m, x^1_m, x^2_m, x^3_m)$ denote the center of the charge distribution. Introduce $\tilde{x}_m = x'_m - x_m$. A Lorentz transformation yields $\tilde{x}^1_r = \tilde{x}^1_m$, $\tilde{x}^2_r = \tilde{x}^2_m$, $\tilde{x}^3_r = \gamma(\tilde{x}^3_m - \beta\tilde{x}^0_m)$, and $\tilde{x}^0_r = \gamma(\tilde{x}^0_m - \beta\tilde{x}^3_m)$ where $\gamma = 1/\sqrt{1-\beta^2}$. Denote this Lorentz transformation by $\tilde{x}_r = L(\tilde{x}_m)$.

In the rest frame, the electron charge e is equal to $\int \rho_r(\tilde{x}_r)\delta(\tilde{x}^0_r)d^4\tilde{x}_r$ where $\rho_r(\tilde{x}_r)$ is the charge density in the rest frame and δ denotes the delta function.[1] So an element of electron charge in the rest frameis given by

$$de_r = \rho_r(\tilde{x}_r)\delta(\gamma(\tilde{x}^0_r)d^4\tilde{x}_r. \tag{1}$$

Label an element of electron charge in the m frame de_m. The equation for de_m is not needed due to invariance of the form factor.[2] The next section is a review of point pair annihilation resulting in a pair of point muons. The third section will study extended electron-extended positron annihilation resulting in an extended muon pair. The S-matrix in the extended lepton theory is the S-matrix of the point lepton theory multiplied by a form factor for the electron-positron and multiplied by a form factor for the muon pair. A short discussion follows.

II. POINT ELECTRON-POSITRON ANNIHILATION

The process where a point electron and a point positron annihilate at spacetime point y producing a photon, which propagates to spacetime point x where it annihilates producing an electron positron pair was studied in a previous paper.[3] The approximate S-matrix for this process (see Eq. 14 in reference 3) is

$$S_{fi} = -\int d^4x\, d^4y\, \bar{\phi}_f(x)(-ie\gamma^{\nu})\phi_I(x) iD_F(x-y)\, \bar{\phi}_F(y)(-ie\gamma_{\nu})\phi_i(y). \tag{2}$$

The above equation can be used for the process where a point electron and the point positron annihilate at spacetime point y producing a photon, which propagates to spacetime point x where it annihilates producing a muon pair. In this case, the initial approximate electron wave function, which has been normalized to unity in a box of volume V, is

$$\phi_i(y) = \sqrt{\frac{m}{E_i V}}\, u_i \exp\left(-ip_i \cdot y\right) \tag{3}$$

where $\hbar$ and c have been set equal to 1, m is the electron rest mass, u_i is a four-component spinor, which depends on the initial spin and on $p_i = (p_i^0 = E_i, p_i^1, p_i^2, p_i^3)$, the initial four-momentum, and γ^ν are the four Dirac matrices, which are labelled by $\nu = 0, 1, 2, 3$. The final μ^- wave function is

$$\phi_f(x) = \sqrt{\frac{M}{E_f V}}\, u_f \exp\left(-ip_f \cdot x\right) \tag{4}$$

where M is the muon rest mass, u_f is a four-component spinor, which depends on the final muon spin and on $p_f = (p_i^0 = E_f, p_f^1, p_f^2, p_f^3)$, the final muon four-momentum.

Let $p_I = (E_I, p_I^1, p_I^2, p_I^3)$ denote the initial positron four-momentum, and let $p_F = (E_F, p_F^1, p_F^2, p_F^3)$ denote the final positively charged muon four-momentum. It is common to treat the positive energy positron, which propagates forward in time, as a negative energy electron, which propagates backward in time.[4] Similarly the positive energy μ^+, which propagates forward in time, is treated as a negatively charged muon, which propagates backward in time. Thus the negative energy electron propagates from the future into the past, and the negative energy muon propagates from the future into the past. To accommodate this, the final four-momentum of this electron is identified as $-p_I = (-E_I, -\mathbf{p}_I)$, and the initial four-momentum of this muon is identified as $-p_F =$

$(-E_F, -\mathbf{p}_F)$. So the final wave function of this electron, which has been normalized to unity is

$$\phi_F(x) = \sqrt{\frac{m}{E_I V}}\, v_I \exp\left(+ip_I \cdot x\right) \tag{5}$$

where v_I is a four component spinor. The initial wave function of this muon is

$$\phi_I(x) = \sqrt{\frac{M}{E_F V}}\, v_F \exp\left(+ip_F \cdot x\right). \tag{6}$$

$D_F(x-y)$ is the photon propagator, which is

$$D_F(x-y) = \int \frac{d^4q}{(2\pi)^4} \exp\left[-iq \cdot (x-y)\right] \frac{-1}{q^2 + i\epsilon} \tag{7}$$

where q is the four-momentum of the photon.

Substituting Eqs. (3), (4), (5), (6), and (7) into Eq. (2) yields

$$\begin{aligned} S_{fi} = {} & \frac{-ie^2 mM(\bar{u}_f \gamma^\nu v_F)(\bar{v}_I \gamma_\nu u_i)}{(2\pi)^4 V^2 \sqrt{E_i E_f E_I E_F}} \\ & \int \frac{d^4x\, d^4y\, d^4q\ \exp[i(p_f + p_F - q)\cdot x]\exp[i(-p_i - p_I + q)\cdot y]}{q^2 + i\epsilon}, \end{aligned} \tag{8}$$

and finally

$$S_{fi} = \frac{-ie^2 mM}{V^2\sqrt{E_i E_f E_I E_F}}(2\pi)^4\delta^4(p_f+p_F-p_i-p_I)\frac{(\bar{u}_f\gamma^\nu v_F)(\bar{v}_I\gamma_\nu u_i)}{(p_i+p_I)^2}. \quad (9)$$

II. EXTENDED ELECTRON-POSITRON SCATTERING

Suppose that the electron initially moves with a speed β in the $+x^3$ direction. This is the m frame. Recall that x'_m is a spacetime electron charge point, x_m is the center of the electron charge distribution and also the argument of the electron wave function. Define $\tilde{x}_m = x'_m - x_m$. Recall that an element of electron charge in the rest frame is labeled de_r, and is given by Eq. (1). Also recall that an element of electron charge in the m frame is labeled de_m.

In the rest frame of the positron, let y'_r be a spacetime charge point, and let y_r be the center of the positron charge. Introduce $\tilde{y}_r = y'_r - y_r$. Take the positron to be moving with a speed β in the $-y^3$ direction in the m frame. The m frame is therefore the center of mass frame. Let y'_m be a spacetime charge point of the positron. Let y_m be the center of the positron charge and the argument of its wave function. Introduce $\tilde{y}_m = y'_m - y_m$. The interaction takes place at charge points, so replace $D_F(x-y)$ in Eq. (10) by $D_F(x'_m - y'_m)$. Also, replace the positive energy electron charge by the four-dimensional integral of de_m, and replace the negative energy charge by the four-dimensional integral of de_m (the speeds are the same). So now the S-matrix S_{FI} is

$$S_{FI} = -\int d^4x_m\, d^4y_m\, \bar{\phi}_f(x_m)(-i\, de_m\gamma^\nu)\phi_I(x_m)$$
$$iD_F(x'_m - y'_m)\, \bar{\phi}_F(y_m)(-i\, de_m\gamma_\nu)\phi_i(y_m). \quad (10)$$

The four-momentum of the photon in the m frame is labeled q_m. Use $D_F(x'_m - y'_m) = D_F(x_m - y_m)\exp(-i\tilde{x}_m \cdot q_m)\exp(+i\tilde{y}_m \cdot q_m)$ to show

$$S_{FI} = -\int d^4x_m\, d^4y_m\, \bar{\phi}_f(x_m)(-ie\gamma^\nu)\phi_I(x_m)iD_F(x_m - y_m)\bar{\phi}_F(y_m)(-ie\gamma_\nu)\phi_i(y_m)$$
$$\int \exp(-i\tilde{x}_m \cdot q'_m)\frac{de_m}{e}\int \exp(+i\tilde{y}_m \cdot q'_m)\frac{de_m}{e}. \quad (11)$$

By the delta function which results when integrating the first integral above, $q_m = p_I + p_i$. Integration leads to

$$S_{FI} = S_{fi}F_e(q)F_\mu(q)\,. \quad (12)$$

where by invariance of the form factor

$$F_\mu(q) = \int \exp(-i\tilde{x}_r \cdot q_r)\frac{\rho_r(\tilde{x}_r)}{e}\delta(\tilde{x}^0_r)d^4\tilde{x}_r, \quad (13)$$

and

$$F_e(q) = \int \exp(+i\tilde{y}_r \cdot q_r)\frac{\rho_r(\tilde{y}_r)}{e}\delta(\tilde{y}_r^0)d^4\tilde{y}_r. \tag{14}$$

Note that q_r is related to q_m by a Lorentz transformation.

III. DISCUSSION

The S-matrix for extended electron-extended positron annihilation resulting in an extended muon pair is the S-matrix for point pair annihilation resulting in a point muon pair multiplied by an electron form factor and multiplied by a muon form factor.

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References

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[2] http://www.electronformfactor.com/Mott-Rutherford Scattering and Beyond

[3] http://www.electronformfactor.com/Bhabha Scattering

[4] R.P. Feynman, *Quantum Electrodynamics*, (Benjamin, 1961), pp. 66-70.