COMPTON SCATTERING

ABSTRACT

The electron charge is considered to be distributed or extended in space. The differential of the electron charge is set equal to a function of electron charge coordinates multiplied by a four-dimensional differential volume element. The four-dimensional integral of this function is required to equal the electron charge in all Lorentz frames. The S-matrix for Compton scattering is the sum of two probability amplitudes. In the extended electron theory, each probability amplitude is the product of the probability amplitude for point electron scattering multiplied by two electron form factors.

I. SUMMARY

The next section calculates the S-matrix for Compton scattering of a point electron. An alternative calculation of the S-matrix for the Compton scattering of a point electron is carried out in the third section. The coordinates for the extended electron are introduced in the fourth section. In the fifth section, the S-matrix for Compton scattering of an extended electron is calculated. A short discussion follows.

Date: August 3, 2013.

II. COMPTON SCATTERING

The calculation of Compton scattering will follow Bjorken and Drell.[1] The probability amplitude for an electron to absorb a photon at spacetime point y, propagate to spacetime point x, and emit a photon is

$$S_{fi1} = \int d^4x d^4y \bar{\phi}_f(x)(-ie\not{A}(q',x))iS_F(x-y)(-ie\not{A}(q,y))\phi_i(y). \quad (1)$$

As is customary, $\hbar$ and c are set equal to one. The initial approximate electron wave function, which has been normalized to unity in a box of volume V, is

$$\phi_i(y) = \sqrt{\frac{m}{E_i V}}\, u_i \exp\left(-ip_i \cdot y\right) \quad (2)$$

where m is the electron mass, E_i is the initial energy, p_i is the initial momentum four-vector, and u_i is a one by four matrix, which depends on the initial momentum and the initial spin. The final electron wave function is

$$\phi_f(x) = \sqrt{\frac{m}{E_f V}}\, u_f \exp\left(-ip_f \cdot x\right) \quad (3)$$

where E_f is the final energy, p_f is the final momentum four-vector, and u_f is a one by four matrix, which depends on the final momentum and the final spin. The vector potential for the absorbed photon is

$$A(q,y) = \epsilon\sqrt{1/(2q^0V)}\exp\left(-iq\cdot y\right) \tag{4}$$

where ϵ is the polarization vector, $q = (q^0, q^1, q^2, q^3)$ is the four-momentum of the absorbed photon, $\not{A}(q,y) = \gamma^\nu\, A_\nu(q,y)$, and $\not{\epsilon} = \gamma^\nu\,\epsilon_\nu$. Here γ^ν are the four Dirac matrices where $(\nu = 0,1,2,3)$. The vector potential of the emitted photon is

$$A(q',x) = \epsilon'\sqrt{1/(2q'^0V)}\exp\left(+iq'\cdot x\right) \tag{5}$$

where ϵ' is its polarization vector, q' is the four-momentum of the emitted photon, $\not{A}(q',x) = \gamma^\mu\, A_\mu(q',x)$, and $\not{\epsilon}' = \gamma^\mu\,\epsilon'_\mu$. The Feynman propagator is

$$S_F(x-y) = \int \exp\left(-ik\cdot(x-y)\right)S_F(k)\frac{d^4k}{(2\pi)^4} \tag{6}$$

where

$$S_F(k) = \frac{\not{k}+m}{k^2-m^2+i\epsilon}\,, \tag{7}$$

and k is the four-momentum of the electron as it propagates from y to x, and $\not{k} = \gamma^\mu k_\mu$.

Substitution of Eqs. (2), (3), (4), (5), (6), and (7) into Eq. (1) leads to

$$S_{fi1} = \frac{-ime^2}{2V^2\sqrt{E_i E_f\, q^0\, q'^0}} \int d^4x\, d^4y\, \exp\left(ix \cdot (p_f + q' - k)\right)$$
$$\exp[i(k - q - p_i) \cdot y] \frac{\bar{u}_f \not{\epsilon}' (\not{k} + m) \not{\epsilon} u_i}{k^2 - m^2 + i\epsilon} \frac{d^4k}{(2\pi)^4}. \quad (8)$$

After integration over x and y,

$$S_{fi1} = \frac{-ime^2}{2V^2\sqrt{E_i E_f\, q^0\, q'^0}} \int (2\pi)^4 \delta^4(p_f + q' - k)$$
$$(2\pi)^4 \delta^4(k - q - p_i) \frac{\bar{u}_f \not{\epsilon}' (\not{k} + m) \not{\epsilon} u_i}{k^2 - m^2 + i\epsilon} \frac{d^4k}{(2\pi)^4}. \quad (9)$$

Integration over k yields

$$S_{fi1} = \frac{-ie^2 m (2\pi)^4 \delta^4(p_f + q' - p_i - q)}{2V^2\sqrt{E_i E_f q^0 q'^0}} \frac{\bar{u}_f \not{\epsilon}' (\not{p}_i + \not{q} + m) \not{\epsilon} u_i}{2p_i \cdot q}. \quad (10)$$

There is a second probability amplitude, S_{fi2}, where the electron at point y emits the photon of four-momentum q', propagates to point x where it absorbs the photon of four-momentum q. The S-matrix for this process is

$$S_{fi2} = \int d^4x\, d^4y\, \bar{\phi}_f(x)(-ie\not{A}(q,x))iS_F(x-y)(-ie\not{A}(q',y))\phi_i(y). \tag{11}$$

Substitution into Eqs. (11) yields

$$S_{fi2} = \frac{-ime^2}{2V^2\sqrt{E_i E_f\, q^0\, q'^0}} \int d^4x\, d^4y\, \exp\left(ix \cdot (p_f - q - k)\right)$$
$$\exp[i(k + q' - p_i) \cdot y] \frac{\bar{u}_f \not{\epsilon}(\not{k} + m)\not{\epsilon}' u_i}{k^2 - m^2 + i\epsilon} \frac{d^4k}{(2\pi)^4}. \tag{12}$$

After integration over x and y,

$$S_{fi2} = \frac{-ime^2}{2V^2\sqrt{E_i E_f\, q^0\, q'^0}} \int (2\pi)^4 \delta^4(p_f - q - k)$$
$$(2\pi)^4 \delta^4(k + q' - p_i) \frac{\bar{u}_f \not{\epsilon}(k + m)\not{\epsilon}' u_i}{k^2 - m^2 + i\epsilon} \frac{d^4k}{(2\pi)^4}. \tag{13}$$

Integration over k yields

$$S_{fi2} = \frac{-ie^2 m(2\pi)^4 \delta^4(p_f + q' - p_i - q)}{2V^2\sqrt{E_i E_f q^0 q'^0}} \frac{\bar{u}_f \not{\epsilon}(\not{p}_i - \not{q}' + m)\not{\epsilon}' u_i}{-2p_i \cdot q'}. \tag{14}$$

The S matrix is $S_{fi} = S_{fi1} + S_{fi2}$.

III. ALTERNATE CALCULATION

In the previous section, integrals of Eqs. (8) and (12) were done over the four components of x and y before integrating over k. The four dimensional integral over components of k should be done first to yield $S_F(x-y)$. The result is something rather unwieldy.[2], so we will instead integrate first over the zeroth component of k followed by integrals over the zeroth component of x and then over the zeroth component of y. Introduce

$$I_S = \int \frac{dk^0}{(2\pi)^4} \exp\left(-ik^0(x^0-y^0)\right) \frac{(\not{k}+m)}{2\sqrt{|\mathbf{k}|^2+m^2}} \left(\frac{1}{k^0-\sqrt{|\mathbf{k}|^2+m^2}+i\epsilon} - \frac{1}{k^0+\sqrt{|\mathbf{k}|^2+m^2}-i\epsilon}\right) \quad (15)$$

Use contour integration to show

$$I_S = -iH(x^0-y^0)\exp[-i\sqrt{|\mathbf{k}|^2+m^2}(x^0-y^0)]\frac{(\gamma^0\sqrt{|\mathbf{k}|^2+m^2})+\gamma^j k_j+m)}{2\sqrt{|\mathbf{k}|^2+m^2}}$$
$$-iH(y^0-x^0)\exp[+i\sqrt{|\mathbf{k}|^2+m^2}(x^0-y^0)]\frac{(-\gamma^0\sqrt{|\mathbf{k}|^2+m^2})+\gamma^j k_j+m)}{2\sqrt{|\mathbf{k}|^2+m^2}}. \quad (16)$$

where $j=1,2,3$ and $H(x)$ is the unit step function. Introduce

$$I_1 = \int H(x^0 - y^0)\exp[-i\sqrt{|\mathbf{k}|^2+m^2}(x^0-y^0) + i(p_f^0 + q'^0)x^0 - i(p_i^0+q^0)y^0]dx^0dy^0, \tag{17}$$

and introduce

$$I_2 = \int H(y^0 - x^0)\exp[+i\sqrt{|\mathbf{k}|^2+m^2}(x^0-y^0) + i(p_f^0 + q'^0)x^0 - i(p_i^0+q^0)y^0]dx^0dy^0, \tag{18}$$

To integrate I_1, set $z^0 = x^0 - y^0$. Then

$$\begin{aligned} I_1 = \int H(z^0)\exp[-i\sqrt{|\mathbf{k}|^2+m^2}(z^0) + i(p_f^0 + q'^0)(z^0+y^0) - i(p_i^0+q^0)y^0]dz^0dy^0 \\ = -2\pi i\delta(p_f^0 + q'^0 - p_i^0 - q^0) \\ \left(\frac{1}{\sqrt{|\mathbf{k}|^2+m^2} - p_f^0 - q'^0} + \pi i\delta(\sqrt{|\mathbf{k}|^2+m^2} - p_f^0 - q'^0)\right) \end{aligned} \tag{19}$$

where the integral over y^0 gives the delta function, and the integral over z^0 is the Fourier transform of a generalized function.[3] By Eq. (21), $\mathbf{k} = \mathbf{p}_f + \mathbf{q}'$. Set the argument of the delta function equal to zero, and arrive at a contradiction. Thus the delta function vanishes since its argument is not zero.

To integrate I_2, set $z^0 = y^0 - x^0$. Then

$$I_2 = \int H(z^0)\exp\left[+i\sqrt{|\mathbf{k}|^2+m^2}(z^0)+i(p_f^0+q'^0)(x^0)-i(p_i^0+q^0)(z^0+x^0)\right]dz^0dx^0$$

$$= -2\pi i\delta(p_f^0+q'^0-p_i^0-q^0)$$

$$\Big(\frac{1}{\sqrt{|\mathbf{k}|^2+m^2}+p_i^0+q^0}+\pi i\delta(\sqrt{|\mathbf{k}|^2+m^2}+p_i^0+q^0)\Big). \quad (20)$$

By Eq. (21), $\mathbf{k} = \mathbf{p}_i + \mathbf{q}$. Set the argument of the delta function equal to zero, and arrive at a contradiction. Thus the delta function vanishes since its argument is not zero. Note that

$$\int d^3x\, d^3y\, \exp\left[-i\mathbf{x}\cdot(\mathbf{p}_f+\mathbf{q}'-\mathbf{k})\right]\exp\left[+i\mathbf{y}\cdot(\mathbf{p}_i+\mathbf{q}-\mathbf{k})\right]$$

$$= (2\pi)^3\delta^3(\mathbf{p}_f+\mathbf{q}'-\mathbf{k})(2\pi)^3\delta^3(-\mathbf{p}_i-\mathbf{q}+\mathbf{k}). \quad (21)$$

Putting these results together yields Eq. (10).

Follow the same order of integration for S_{fi2}, which is given by Eq. (11), and recover Eq. (14).

This suggests, but does not prove that the multiple integrals done in any order in the Compton scattering S-matix calculation will give the correct result.

IV. EXTENDED ELECTRON COORDINATES

Pick the electron to be initially at rest. Let $y'_r = (y^0_r, y^1_r, y^2_r, y^3_r)$ denote a spacetime electron charge point. Let the spacetime point $y_r = (y^0_r, y^1_r, y^2_r, y^3_r)$ denote the argument of the wave function and the center of the electron charge distribution. Define $\tilde{y}_r = y'_r - y_r$. In the rest frame, the electron charge e is equal to $\int \rho_r(\tilde{y}_r)\delta(\tilde{y}^0_r)d^4\tilde{y}_r$ where $\rho_r(\tilde{y}_r)$ is the charge density in the rest frame and δ denotes the delta function.[4] So an element of charge in the rest frame is given by

$$de_r = \rho_r(\tilde{y}_r)\delta(\tilde{y}^0_r)\, d^4\tilde{y}_r. \tag{22}$$

Take the photon, which has four-momentum q, to be initially moving in the $+y^3$ direction. The photon is absorbed at an electron charge point y'_r. The electron now propagates in the $+y^3$ direction with a speed β. The electron then emits a photon of four-momentum q' from the moving spacetime charge point $x'_m = (x'^0_m, x'^1_m, x'^2_m, x'^3_m)$. Denote the center of the electron charge distribution and the argument of the wave function by $x_m = (x^0_m, x^1_m, x^2_m, x^3_m)$. Introduce $\tilde{x}_m = x'_m - x_m$.

Transform from the m frame to the rest frame where $x'_r = (x'^0_r, x'^1_r, x'^2_r, x'^3_r)$ denotes a spacetime charge point of the electron charge distribution, and $x_r = (x^0_r, x^1_r, x^2_r, x^3_r)$ denotes the center of the charge distribution. The charge distribution of the electron is assumed to have a well-defined center, which is identified as the argument of the wave function. The shape of the charge distribution depends on the motion of the charge, and is assumed to be unaffected by any interaction. Introduce $\tilde{x} = x' - x$. A Lorentz transformation yields $\tilde{x}^1_r = \tilde{x}^1_m$, $\tilde{x}^2_r = \tilde{x}^2_m$, $\tilde{x}^3_r = \gamma(\tilde{x}^3_m - \beta\tilde{x}^0_m)$, and $\tilde{x}^0_r = \gamma(\tilde{x}^0_m - \beta\tilde{x}^3_m)$ where $\gamma = 1/\sqrt{1-\beta^2}$. In the m frame, the electron charge [5] will be denoted by de_m where

$$e = \int de_m = \int \rho_r(\tilde{x}_m)\, \delta[\gamma(\tilde{x}^0_m - \beta\tilde{x}^3_m)]\, d^4\tilde{x}_m \qquad (23)$$

V. COMPTON SCATTERING OF EXTENDED ELECTRONS

Replace the electron charge of the electron at rest by the four-dimensional volume integral of de_r. Replace the electron charge of the moving electron by the four-dimensional volume integral of de_m. Since the photon is emitted and absorbed at a charge point, replace $A_\nu(q, y)$ by $A_\nu(q, y'_r)$, and replace $A_\mu(q', x)$ by $A_\mu(q', x'_m)$ in the first probability amplitude. Then,

$$S_{FI1} = \int d^4x_m\, d^4y_r\, \bar{\phi}_f(x_m)(-ie\not{A}(q', x'_m))\frac{de_m}{e}$$
$$iS_F(x_m - y_r)(-ie\not{A}(q, y'_r))\,\frac{de_r}{e}\,\phi_i(y_r) =$$
$$-ie^2 \int d^4x_m\, d^4y_r\, \bar{\phi}_f(x_m)\not{A}(q', x_m)S_F(x_m - y_r)\not{A}(q, y'_r)\phi_i(y_r)$$
$$\int \exp(iq' \cdot \tilde{x}_m)\frac{de_m}{e} \int \exp(-iq \cdot \tilde{y}_r)\frac{de_r}{e}. \quad (24)$$

The first integral is identified as the first probability amplitude for point electron Compton scattering (see Eq. (1)), so

$$S_{FI1} = S_{fi1}F_m(q')F_r(q). \quad (25)$$

The spacetime points x and y in Eq. (1) refer to an arbitrary frame of reference, which includes the frame of reference used in this section. By Eq. (22)

$$F_r(q) = \int \exp(iq \cdot \tilde{y}_r)\frac{de_r}{e} = \int \exp(iq \cdot \tilde{y}_r)\frac{\rho_r(\tilde{y}_r)}{e}\delta(\tilde{y}_r^0)\, d^4\tilde{y}_r\,. \quad (26)$$

By invariance of the form factor

$$F_m(q') = \int \exp{(iq' \cdot \tilde{x}_m)} \frac{de_m}{e} = \int \exp{(iq'_r \cdot \tilde{x}_r)} \frac{de_r}{e} =$$
$$\int \exp{(iq'_r \cdot \tilde{x}_r)} \frac{\rho_r(\tilde{x}_r)}{e} \delta(\tilde{x}^0_r)\, d^4\tilde{x}_r\,. \quad (27)$$

where $q'^0_r = \gamma_1(q'^0 + \beta_1 q'^3), q'^1_r = q'^1, q'^2_r = q'^2$, and $q'^3_r = \gamma_1(q'^3 + \beta_1 q'^0)$ and $\gamma_1 = 1/\sqrt{1-\beta_1^2}$. Notice that β_1, the speed at which the electron propagates between y and x, appears. The propagator speed is $|\mathbf{k}|/k^0$. It is now important to integrate over k first. The incorrect propagator speed is calculated by integrating over x and y before integrating over k. So integrate over k^0 first and find that the propagator speed β_1 is $|\mathbf{k}|/\sqrt{|\mathbf{k}|^2 + m^2}$. Then the correct propagator speed is identified as $|\mathbf{q}|/\sqrt{|\mathbf{q}|^2 + m^2}$ since $\mathbf{k} = \mathbf{q}$ when the electron is initially at rest.

The second probability amplitude is given by

$$S_{FI2} = \int d^4x_m\, d^4y_r\, \bar{\phi}_f(x_m)(-ie\not{A}(q, x'_m)) \frac{de_m}{e}$$
$$iS_F(x_m - y_r)(-ie\not{A}(q', y'_r))\, \frac{de_r}{e}\, \phi_i(y_r) =$$
$$-ie^2 \int d^4x_m\, d^4y_r\, \bar{\phi}_f(x_m)\not{A}(q, x_m) S_F(x_m - y_r)\not{A}(q', y_r)\phi_i(y_r)$$
$$\int \exp{(iq \cdot \tilde{x}_m)} \frac{de_m}{e} \int \exp{(-iq \cdot \tilde{y}_r)} \frac{de_r}{e}. \quad (28)$$

The first integral is identified as the second probability amplitude for point electron Compton scattering (see Eq. (11)), so

$$S_{FI2} = S_{fi2} F_m(q) F_r(q') \tag{29}$$

where

$$F_r(q') = \int \exp{(iq' \cdot \tilde{y}_r)} \frac{de_r}{e} = \int \exp{(iq' \cdot \tilde{y}_r)} \rho_r(\tilde{y}_r) \delta(\tilde{y}_r^0)\, d^4\tilde{y}_r\,. \tag{30}$$

It is convenient to pick $\mathbf{q}'$ to be in x^3 direction. Then by invariance of the form factor,

$$F_m(q) = \int \exp{(iq \cdot \tilde{x}_m)} \frac{de_m}{e} = \int \exp{(iq_r \cdot \tilde{x}_r)} \frac{de_r}{e} = \\ \int \exp{(iq_r \cdot \tilde{x}_r)} \frac{\rho_r(\tilde{x}_r)}{e} \delta(\tilde{x}_r^0)\, d^4\tilde{x}_r\,. \tag{31}$$

where $q_r^0 = \gamma_2(q^0 - \beta_2 q^3), q_r^1 = q^1, q_r^2 = q^2$, and $q_r^3 = \gamma_2(q^3 - \beta_2 q^0)$ and $\gamma_2 = 1/\sqrt{1-\beta_2^2}$. Notice that β_2, the speed at which the electron propagates between y and x, appears. The propagator speed is $|\mathbf{k}|/k^0$. Again it is now important to integrate over k^0 first. The incorrect propagator speed is calculated by integrating over x and y before integrating over k. The correct propagator speed is $|\mathbf{q}'|/\sqrt{|\mathbf{q}'|^2 + m^2}$ since $\mathbf{k} = -\mathbf{q}'$.

VI. DISCUSSION

Since the propagator speed enters the extended electron theory, it is important to integrate over k^0 first. Integration over x and y before integrating over k leads to an incorrect propagator speed.

The S-matrix for Compton scattering contains two probability amplitudes. In the extended electron theory, each probability amplitude is the probability amplitude for the point electron Compton scattering, times two electron form factors. These electron form factors should be determined from experiment.

ACKNOWLEDGMENTS

I thank Ben for his many contributions to this paper.

References

[1] J.D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill, New York, 1964), pp. 127-128.

[2] R.P. Feynman, *Quantum Electrodynamics*, (Benjamin, 1961), pp. 84-85.

[3] D.S. Jones, *Geralized Functions* (McGraw-Hill, Berkshire, England 1966), p. 99.

[4] S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity* (John Wiley & Sons, Inc. New York, 1972), pp. 40-41.

[5] http://www.electronformfactor.com/Mott-Rutherford Scattering and Beyond