ELECTRON-PION SCATTERING II

Abstract

The electron charge is considered to be distributed or extended in space. The differential of the electron charge is set equal to a function of electron charge coordinates multiplied by a four-dimensional differential volume element. The four-dimensional integral of this function is required to equal the electron charge in all Lorentz frames. The pion current is modified by taking into account virtual strong interactions. In addition, the pion current is considered to be extended in space like the electron. The result is that for extended electron-pion scattering, the S-matrix is the S-matrix for point electron-point pion scattering multiplied by an electron form factor and multiplied by a pion form factor, which consists of two terms.

I. POINT ELECTRON-PION SCATTERING

The calculation of the S-matrix will follow Aitchison and Hey[1] with a little of Borken and Drell thrown in.[2] The probability amplitude for an electron at spacetime point x to exchange a photon of four-momentum q with a pion at spacetime point y is

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$$S_{fi} = -i \int d^4x\, d^4y\, \bar{\phi}_f(x)(-e\gamma_\mu)\phi_i(x) D_F(x-y) j^\mu(y) \tag{1}$$

where $-e$ is the electron charge, $\phi_f(x)$ is the final electron wave function, $\gamma_\mu(\mu = 0, 1, 2, 3)$ are the Dirac matrices, $\phi_i(x)$ is the initial electron wave function, and $\bar{\phi}_f = \phi_f^\dagger \gamma^0$. $D_F(x-y)$ is the photon propagator. For a positively charged pion, the pion current $j^\mu(y)$ is

$$j^\mu(y) = ie\big(\phi_F^*(y)\partial_\nu\phi_I(y) - (\partial_\nu\phi_F^*(y))\phi_I(y)\big) \tag{2}$$

where $\phi_F(y)$ is the final pion wave function, and $\phi_I(y)$ is the initial pion wave function. Each wave function is normalized to equal two times its energy in a box of volume V:

$$\phi_i(x) = \sqrt{\frac{1}{V}}\, u_i \exp\left(-ip_i \cdot x\right) \tag{3}$$

where u_i is a four-component spinor, which depends on the initial spin and on the initial four-momentum p_i,

$$\phi_f(x) = \sqrt{\frac{1}{V}}\, u_f \exp\left(-ip_f \cdot x\right) \tag{4}$$

where u_f is a four component spinor, which depends on the final spin and on the final four-momentum p_f,

$$\phi_I(y) = \sqrt{\frac{1}{V}} \exp(-ip_I \cdot y) \tag{5}$$

where p_I is the initial pion four-momentum,

$$\phi_F(y) = \sqrt{\frac{1}{V}} \exp(-ip_F \cdot y) \tag{6}$$

where p_F is the final pion four-momentum. Substitute Eqs. (5), and (6) into Eq. (2), and get

$$j^\mu(y) = \frac{e}{V}(p_F + p_I)^\mu \, \exp\left(i(p_F - p_I) \cdot y\right). \tag{7}$$

The photon propagator is given by

$$D_F(x-y) = -\int \frac{\exp\left[-i(x-y)\cdot q\right]}{q^2 + i\epsilon} \frac{d^4q}{(2\pi)^4} \tag{8}$$

Substitute Eqs. (3), (4), (7), and (8) into Eq. (1), and get

$$S_{fi} = +i\frac{e^2}{V^2}(\bar{u}_f \gamma_\mu u_i)(p_F + p_I)^\mu \int d^4x \, d^4y \, \exp\left(i(p_f - p_i)\cdot x\right)$$
$$\exp\left(-iq\cdot(x-y)\right) \exp\left(i(p_f - p_I)\cdot y\right) \frac{-1}{q^2 + i\epsilon} \frac{d^4q}{(2\pi)^4} \tag{9}$$

Integrate over the four components of x and y and get

$$S_{fi} = +i\frac{e^2}{V^2}(\bar{u}_f\gamma_\mu u_i)(p_F + p_I)^\mu \int (2\pi)^4\delta^4(p_f - p_i - q)$$
$$(2\pi)^4\delta^4(p_F - p_I + q)\frac{-1}{q^2 + i\epsilon}\frac{d^4q}{(2\pi)^4} \quad (10)$$

Finally

$$S_{fi} = -i\frac{e^2}{V^2}(\bar{u}_f\gamma_\mu u_i)(p_I + p_F)^\mu \frac{(2\pi)^4\delta^4(p_F + p_f - p_I - p_i)}{q^2}\,. \quad (11)$$

where $q = p_f - p_i = p_I - p_F$

II. EXTENDED ELECTRON COORDINATES

Let $x_r'^\mu = (x_r'^0, x_r'^1, x_r'^2, x_r'^3)$ denote a spacetime charge point, in the rest frame of an electron charge distribution, and let $x_r^\mu = (x_r^0, x_r^1, x_r^2, x_r^3)$ denote the center of the charge distribution. The charge distribution of the electron is assumed to have a well-defined center, which is identified as the argument of the wave function. The shape of the charge distribution depends on the motion of the charge, and is assumed to be unaffected by any interaction. Sometimes the superscript will be omitted, and we will write $x_r' = (x_r'^0, x_r'^1, x_r'^2, x_r'^3)$ and $x_r = (x_r^0, x_r^1, x_r^2, x_r^3)$. Introduce $\tilde{x}_r = x_r' - x_r$ or equivalently $\tilde{x}_r^\mu = x_r'^\mu - x_r^\mu$. In a frame of reference in which the charge distribution moves with a speed β in the

$+x^3$ direction, let $x'_m = (x'^0_m, x'^1_m, x'^2_m, x'^3_m)$ denote a spacetime charge point, and let $x_m = (x^0_m, x^1_m, x^2_m, x^3_m)$ denote the center of the charge distribution. Introduce $\tilde{x}_m = x'_m - x_m$. A Lorentz transformation yields $\tilde{x}^1_r = \tilde{x}^1_m$, $\tilde{x}^2_r = \tilde{x}^2_m$, $\tilde{x}^3_r = \gamma(\tilde{x}^3_m - \beta\tilde{x}^0_m)$, and $\tilde{x}^0_r = \gamma(\tilde{x}^0_m - \beta\tilde{x}^3_m)$ where $\gamma = 1/\sqrt{1-\beta^2}$.

In the rest frame, the electron charge e is equal to $\int \rho_r(\tilde{x}_r)\delta(\tilde{x}^0_r)d^4\tilde{x}_r$ where $\rho_r(\tilde{x}_r)$ is the charge density in the rest frame and δ denotes the delta function.[3] So an element of charge in the rest frame is given by

$$de_r = \rho_r(\tilde{x}_r)\delta(\tilde{x}^0_r)\, d^4\tilde{x}_r. \tag{12}$$

In the m frame, the differential electric charge will be denoted by de_m and is given by

$$de_m = \rho_r(\tilde{x}_m)\delta(\gamma(\tilde{x}^0_m - \beta\tilde{x}^3_m)\, d^4\tilde{x}_m.^4 \tag{13}$$

Similar definitions can be given for the pion coordinates. For the pion initially moving with a speed β_p in the $-y^3$ direction,

$$de_{mp} = \rho_{rp}(\tilde{y}_m)\delta(\gamma_p(\tilde{y}^0_m + \beta_p\tilde{y}^3_m)\, d^4\tilde{y}_m. \tag{14}$$

III. EXTENDED ELECTRON-PION SCATTERING

Electron size is taken into account by replacing the electron charge by the four-dimensional volume integral of de_m and replacing $D_F(x-y)$ by $D_F(x'-y)$. Pion size is taken into account by replacing the pion charge by the four-dimensional volume integral of de_{pm} and replacing $D_F(x'-y)$ by $D_F(x'-y')$. Application of Lorentz invariance and conservation of charge to virtual strong interactions effects at the pion vertex, suggest that the point pion current $e(p_F+p_I)$ should be replaced by $F(q^2)e(p_F+p_I)$ where the invariant $F(q^2)$ is referred to as the pion form factor.[1] As a result S_{fi} is replaced by

$$S_{FI} = -i \int d^4x\, d^4y\, \bar{\phi}_f(x)(de_m \gamma_\mu)\phi_i(x) D_F(x'-y')$$

$$(p_F+p_I)^\mu\, de_{mp} F(q^2)\phi_F^*(y)\phi_I(y). \quad (15)$$

Substitute Eqs. (3), (4), (7), and (8) into Eq. (15), and get

$$S_{FI} = +i\frac{e^2}{V^2}(\bar{u}_f\gamma_\mu u_i)(p_F+p_I)^\mu \int d^4x\, d^4y\, \exp\left(i(p_f-p_i)\cdot x\right)$$

$$\exp\left(-iq\cdot(x-y)\right)\exp\left(i(p_F-p_I)\cdot y\right)\frac{-1}{q^2+i\epsilon}\frac{d^4q}{(2\pi)^4}$$

$$F(q^2)\int \exp\left(-iq\cdot\tilde{x}\right)\frac{de_m}{-e}\int \exp\left(+iq\cdot\tilde{y}\right)\frac{de_{mp}}{+e}. \quad (16)$$

S_{FI} can be written

$$S_{FI} = S_{fi} F(q^2) F_e(q) F_p(q) \tag{17}$$

where the electron form factor $F_e(q)$ is given by

$$F_e(q) = \int \exp\left(-iq \cdot \tilde{x}\right) \frac{de_m}{-e}, \tag{18}$$

and the additional pion form factor is given by

$$F_p(q) = \int \exp\left(+iq \cdot \tilde{y}\right) \frac{de_{mp}}{+e}. \tag{19}$$

where the coordinates $\tilde{x}, \tilde{y}$ and the photon four-momentum q are to be measured in the m frame (the lab frame). However the form factors are invariant, so it will be convenient to calculate them in the rest frame. So

$$F_e(q) = \int \exp\left(-iq_r \cdot \tilde{x}_r\right) \frac{\rho_r(\tilde{x}_r)}{e} \delta(\tilde{x}_r^0)\, d^4\tilde{x}_r, \tag{20}$$

and

$$F_p(q) = \int \exp\left(+iq_{rp} \cdot \tilde{y}_r\right) \frac{\rho_{rp}(\tilde{y}_r)}{+e} \delta(\tilde{y}_r^0) d^4\tilde{y}_r. \tag{21}$$

To get a rough idea of how size and structure affect scattering, pick- the electron charge to be uniformly distributed on a spherical shell oradius a in the rest frame. So

$$\rho_r(\tilde{x}_r) = \frac{-e}{4\pi a^2}\,\delta(\sqrt{(\tilde{x}_r^1)^2 + (\tilde{x}_r^2)^2 + (\tilde{x}_r)^2} - a). \tag{22}$$

In spherical coordinates, $(\tilde{r}_r)^2 = (\tilde{x}_r^1)^2 + (\tilde{x}_r^2)^2 + (\tilde{x}_r^3)^2$, so that $d^3\tilde{x}_r = (\tilde{r}_r)^2 \sin\tilde{\theta}_r\, d\tilde{\theta}_r\, d\tilde{\phi}_r\, d\tilde{r}_r$. Then

$$F(q) = \int \frac{\delta(|\tilde{\mathbf{r}}| - a)}{4\pi a^2} \exp\left(+i\mathbf{q}_r \cdot \tilde{\mathbf{r}}\right)\hat{r}^2 \sin\tilde{\theta}\, d\tilde{\theta}\, d\tilde{\phi}\, d\tilde{r} = j_0(|\mathbf{q}_r|a) \tag{23}$$

where $|\mathbf{q}_r|^2 = (q_r^1)^2 + (q_r^2)^2 + (q_r^3)^2 = (q_m^1)^2 + (q_m^2)^2 + \gamma^2(q_m^3 - \beta q_m^0)^2$ and j_0 is the spherical Bessel function of order 0. Here $|\mathbf{q}_r|$ is expressed in terms of the components of $q = p_f - p_i$, since it is p_f and p_i which are measured in the m or lab frame.

Similarly, pick the pion charge distribution to be a spherical shell of radius a_p in the rest frame. Then,

$$\rho_{rp}(\tilde{y}_r) = \frac{e}{4\pi a_p^2}\,\delta(\sqrt{(\tilde{y}_r^1)^2 + (\tilde{y}_r^2)^2 + (\tilde{y}_r^3)^2} - a_p). \tag{24}$$

Introduce spherical coordinates again. Proceed as with the electron, and find $F_p(q) = j_0(|\mathbf{q}_{rp}|a_p)$ where $|\mathbf{q}_{rp}|^2 = (q_m^1)^2 + (q_m^2)^2 + \gamma_p^2(q_m^3 + \beta_p q_m^0)^2$.

For arbitrary charge distributions, it appears that the form factors are functions of $|\mathbf{q_r}|$ and $|\mathbf{q_{rp}}|$ times the appropriate radius. So in general

$$S_{FI} = S_{fi} F(q_{rp}^2) F_e(|\mathbf{q_r}|a) F_p(|\mathbf{q_{rp}}|a_p)\,. \tag{25}$$

IV. DISCUSSION

The S-matrix for both the electron and pion with size is the S-matrix for point electron-point pion scattering multiplied by an electron form factor and multiplied by two pion form factors. These form factors should be determined by experiment.

The ideas here can be applied to electron-proton scattering. The result will be to modify the Rosenbluth formula.

ACKNOWLEDGMENTS

References

[1] I Aitchison and A Hey, *Gauge Theories in Particle Physics* (Adam Hilger Ltd, Bristol, 1982), pp. 62-74.

[2] J.D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill, New York, 1964), Chapter 7.

[3] S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity* (John Wiley & Sons, Inc. New York, 1972), pp. 40-41.

[4] http://www.electronformfactor.com/Mott-Rutherford Scattering and Beyond.