ELECTRON-PROTON SCATTERING II

ABSTRACT

The electron charge is considered to be distributed or extended in space. The differential of the electron charge is set equal to a function of electron charge coordinates multiplied by a four-dimensional differential volume element. The four-dimensional integral of this function is required to equal the electron charge in all Lorentz frames. The proton is also considered to be extended in space. The extension and structure of the proton can be written in terms of three proton form factors. The S-matrix for the extended electron and extended proton is calculated. The result is that the S-matrix contains four form factors.

I. POINT ELECTRON- PROTON SCATTERING

The S-matrix for point electron-proton scattering is [1,2]

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$$S_{fip} = \int d^4x\, d^4y\, \bar{\phi}_f(x)(-ie\gamma_\mu))\, \phi_i(x) iD_F(x-y)$$
$$\phi_F(y)(+ie)\Big(\gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2M}F_2(q^2)\Big)\phi_I(y). \quad (1)$$

The two proton form factors, $F_1(q^2)$ and $F_2(q^2)$, are due to virtual strong interactions. Lorentz invariance and charge conservation determine the mathematical form of line two in Eq. (1). For the point proton, the second line in Eq. (1) would read $\bar{\phi}_F(y)(ie\gamma^\mu)\phi_I(y)$.

The initial exact electron wave function is approximated by the plane wave solution to the Dirac equation. The plane wave solution, which is normalized to unity in a box of volume V, is

$$\phi_i(x) = \sqrt{\frac{m}{E_i V}}\, u_i \exp\left(-ip_i \cdot x\right) \quad (2)$$

where $\hbar$ and c have been set equal to 1, m is the electron rest mass, u_i is a four-component spinor, which depends on the initial spin and on $p_i = (p_i^0 = E_i, p_i^1, p_i^2, p_i^3)$, the initial four-momentum, and γ^μ are the four Dirac matrices, which are labelled by $\mu = 0, 1, 2, 3$. The final electron wave function is

$$\phi_f(x) = \sqrt{\frac{m}{E_f V}}\, u_f \exp\left(-ip_f \cdot x\right), \tag{3}$$

where p_f is the final electron four-momentum, E_f is the final electron energy, u_f is the final electron spinor, and $\bar{\phi}_f = \phi_f^\dagger \gamma^0$. The photon propagator is

$$D_F(x-y) = \int \frac{d^4q}{(2\pi)^4} \exp\left[-iq\cdot(x-y)\right] \frac{-1}{q^2+i\epsilon} \tag{4}$$

where q is the four-momentum of the photon, and the spacetime point y is the the argument of the proton wave function. The initial approximate proton wave function is

$$\phi_I(y) = \sqrt{\frac{M}{E_I V}}\, u_I \exp(-ip_I \cdot y) \tag{5}$$

where M is the proton mass, u_I is a four component spinor, which depends on the initial proton spin and on $p_I = (p_I^0 = E_I, p_I^1, p_I^2, p_I^3)$, the initial proton four-momentum. The final proton wave function is

$$\phi_F(y) = \sqrt{\frac{M}{E_F V}}\, u_F \exp(-ip_F \cdot y) \tag{6}$$

where p_F is the final proton momentum four-vector,E_F is the final proton energy, and u_F is the final four-component spinor of the proton, and $\sigma^{\mu\nu} = i[\gamma^\mu, \gamma^\nu]/2$. Substituting Eqs. (2), (3), (4), (5), and (6) into Eq. (1) yields

$$S_{fip} = \frac{+ie^2 mM(\bar{u}_f \gamma^\mu u_i)(\bar{u}_F \gamma_\mu u_I)}{(2\pi)^4 V^2 \sqrt{E_i E_f E_I E_F}} \int \Big(\gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu} q_\nu}{2M} F_2(q^2) \Big) \frac{d^4x\, d^4y\, d^4q \ \exp[i(p_f - p_i - q)\cdot x] \exp[i(p_F - p_I + q)\cdot y]}{q^2 + i\epsilon} \tag{7}$$

Perform the following integrations:

$$\int \exp(i(p_f - p_i - q)\cdot x) d^4x = (2\pi)^4 \delta^4(p_f - p_i - q); \tag{8}$$

$$\int \exp(i(p_F - p_I + q)\cdot y) d^4y = (2\pi)^4 \delta^4(p_F - p_I + q); \tag{9}$$

$$\int \delta^4(p_f - p_i - q)\delta^4(p_F - p_I + q) \frac{d^4q}{q^2 + i\epsilon} = \frac{\delta^4(p_F + p_f - p_I - p_i)}{(p_f - p_i)^2}; \tag{10}$$

and find

$$S_{fip} = \frac{+ie^2 mM}{V^2 \sqrt{E_i E_f E_I E_F}} \frac{(2\pi)^4 \delta^4(p_f + p_F - p_i - p_I)}{(p_f - p_i)^2} (\bar{u}_f \gamma_\mu u_i) \Big(\bar{u}_F \big(\gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu} q_\nu}{2M} F_2(q^2) \big) u_I \Big). \tag{11}$$

where $q = p_f - p_i$.

II. ELECTRON AND PROTON COORDINATES

In the rest frame of a proton, let $y_r'^{\mu} = (y_r'^0, y_r'^1, y_r'^2, y_r'^3)$ denote a proton charge point, and let $y_r^{\mu} = (y_r^0, y_r^1, y_r^2, y_r^3)$ denote the center of the proton charge distribution and also the argument of the proton wave function. Sometimes the superscript on the four-vector (not the components) will be omitted, and we will write $y_r' = (y_r'^0, y_r'^1, y_r'^2, y_r'^3)$ and $y_r = (y_r^0, y_r^1, y_r^2, y_r^3)$. Introduce $\tilde{y}_r = y_r' - y_r$ or the equivalent $\tilde{y}_r^{\mu} = y_r'^{\mu} - y_r^{\mu}$. In the rest frame, the proton charge $-e$ is equal to $\int \rho_{rp}(\tilde{y}_r)\delta(\tilde{y}_r^0)d^4\tilde{y}_r$ where $\rho_{rp}(\tilde{y}_r)$ is the proton charge density in the rest frame and δ denotes the delta function.[3] Thus an element of proton charge is given by

$$de_{rp} = \rho_{rp}(\tilde{y}_r)\delta(\tilde{y}_r^0)d^4\tilde{y}_r \tag{12}$$

In the rest frame of an electron, let $x_r' = (x_r'^0, x_r'^1, x_r'^2, x_r'^3)$ denote a spacetime charge point, and let $x_r = (x_r^0, x_r^1, x_r^2, x_r^3)$ denote the center of the electron charge distribution and also the argument of the electron wave function. Introduce $\tilde{x}_r = x_r' - x_r$. In a frame of reference in which the electron charge distribution moves with a speed β in the $+x^3$ direction, let $x'_m = (x'^0_m, x'^1_m, x'^2_m, x'^3_m)$ denote a spacetime charge point, and let $x_m = (x^0_m, x^1_m, x^2_m, x^3_m)$ denote the center of the charge distribution. Introduce $\tilde{x}_m = x'_m - x_m$. A Lorentz transformation yields $\tilde{x}^1_r = \tilde{x}^1_m$, $\tilde{x}^2_r = \tilde{x}^2_m$, $\tilde{x}^3_r = \gamma(\tilde{x}^3_m - \beta\tilde{x}^0_m)$, and $\tilde{x}^0_r = \gamma(\tilde{x}^0_m - \beta\tilde{x}^3_m)$ where $\gamma = 1/\sqrt{1-\beta^2}$. In the rest frame, the electron charge e is equal to $\int \rho_r(\tilde{x}_r)\delta(\tilde{x}^0_r)d^4\tilde{x}_r$ where $\rho_r(\tilde{x}_r)$ is the charge density in the rest frame[3]. The electric charge e is equal to $\int \rho_r(\tilde{x}_m)\delta[\gamma(\tilde{x}^0_m - \beta\tilde{x}^3_m)]d^4\tilde{x}_m$ in the m frame.[4] So an element of electron charge de_m in the m frame is given by

$$de_m = \rho_r(\tilde{x}_m)\delta[\gamma(\tilde{x}^0_m - \beta\tilde{x}^3_m)]d^4\tilde{x}_m. \tag{13}$$

III. EXTENDED ELECTRON-EXTENDED PROTON SCATTERING

Take the electron to be initially moving with a speed β in the $+x^3$ direction and the proton to be initially at rest. This is the m frame, which is also called the lab frame. Eq. (1) is now modified to take into account the spatial charge distributions of the electron and the proton. Since the interaction takes place at charge points, $D_F(x-y)$ is replaced by $D_F(x'_m - y'_r)$. In addition, the electron charge is replaced by the four-dimensional integral of de_m, and the proton charge is replaced by the four-dimensional integral of de_{rp}. Then S_{fip} is replaced by S_{fie}, the S-matrix for extended electron-extended proton scattering where

$$S_{fie} = \int d^4x_m\, d^4y_r\, \bar{\phi}_f(x_m)(-ide_m\gamma_\mu))\,\phi_i(x_m) iD_F(x'_m - y'_r)$$
$$\phi_F(y_r)(+ide_{rp})\Big(\gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2M}F_2(q^2)\Big)\phi_I(y_r). \quad (14)$$

Upon making the above substitutions

$$S_{fie} = \int d^4x_m\, d^4y_r\, \bar{\phi}_f(x_m)(-ie\gamma_\mu))\,\phi_i(x_m) iD_F(x_m - y_r)$$
$$\phi_F(y_r)(+ie)\Big(\gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2M}F_2(q^2)\Big)\phi_I(y_r)$$
$$\exp\left(-iq\cdot\tilde{x}_m\right)\frac{\rho_r(\tilde{x}_m)}{e}\delta[\gamma(\tilde{x}^0_m - \beta\tilde{x}^3_m)]d^4\tilde{x}_m$$
$$\exp\left(+iq\cdot\tilde{y}_r\right)\frac{\rho_{rp}(\tilde{y}_r)}{-e}\delta(\tilde{y}^0_r)d^4\tilde{y}_r. \quad (15)$$

This equation can be written

$$S_{fie} = S_{fip}F_e(q)F_p(q). \quad (16)$$

where the electron form factor $F_e(q)$ is given by

$$F_e(q) = \int \exp\left(-iq\cdot\tilde{x}_m\right)\frac{\rho_r(\tilde{x}_m)}{e}\delta[\gamma(\tilde{x}^0_m - \beta\tilde{x}^3_m)]d^4\tilde{x}_m, \quad (17)$$

and the proton form factor $F_p(q)$ is given by

$$F_p(q) = \int \exp(+iq \cdot \tilde{y}_r) \frac{\rho_{rp}(\tilde{y}_r)}{-e} \delta(\tilde{y}_r^0) d^4 \tilde{y}_r. \tag{18}$$

To get a rough idea of how proton size affects scattering, pick the proton charge to be uniformly distributed on a spherical shell radius a_p in the rest frame. So

$$\rho_{rp}(\tilde{y}_r) = \frac{-e}{4\pi a_p^2} \delta(\sqrt{(\tilde{y}_r^1)^2 + (\tilde{y}_r^2)^2 + (\tilde{y}_r)^2} - a_p). \tag{19}$$

In spherical coordinates, $(\tilde{r}_r)^2 = (\tilde{y}_r^1)^2 + (\tilde{y}_r^2)^2 + (\tilde{y}_r^3)^2$, so that $d^3\tilde{y}_r = (\tilde{r}_r)^2 \sin\tilde{\theta}_r \, d\tilde{\theta}_r \, d\tilde{\phi}_r \, d\tilde{r}_r$. Then

$$F_p(q) = \int \frac{\delta(|\tilde{\mathbf{r}}_{\mathbf{r}}| - a_p)}{4\pi a_p^2} \exp(+i\mathbf{q} \cdot \tilde{\mathbf{r}}_{\mathbf{r}}) \hat{r}_r^2 \sin\tilde{\theta}_r \, d\tilde{\theta}_r \, d\tilde{\phi}_r \, d\tilde{r}_r = j_0(|\mathbf{q}|a_p) \tag{20}$$

where j_0 is the spherical Bessel function of order 0.

Pick the electron charge to be distributed uniformly on a spherical shell of radius a in the rest frame. The form factor is invariant, so

$$F_e(q) = \int \exp(-iq_r \cdot \tilde{x}_r) \frac{\rho_r(\tilde{x}_r)}{e} \delta(\tilde{x}^0) \, d^4\tilde{x}_r = j_0(|\mathbf{q}_r|a) \tag{21}$$

where by a Lorentz transformation $|\mathbf{q}_r|^2 = (q^1)^2 + (q^2)^2 + \gamma^2(q^3 - \beta q^0)^2$. Thus

$$S_{fie} = S_{fip} j_0(|\mathbf{q}|a_p) j_0(|\mathbf{q}_r|a). \tag{22}$$

The preceding calculations suggest that for unknown charge distributions, the proton form factor is a function of $|\mathbf{q}|a_p$, and the electron form factor is a function of $|\mathbf{q}_r|a$. So for an unknown charge distributions

$$S_{fie} = S_{fip} F(|\mathbf{q}|a_p) F(|\mathbf{q}_r|a). \tag{23}$$

IV. DISCUSSION

To recapitulate, the form factors $F_1(q^2)$ and $F_2(q^2)$ are due to virtual strong interactions. The form factors $F(|\mathbf{q}|a_p)$ and $F(|\mathbf{q}_r|a)$ are due to the extent of the proton charge and the electron charge in space. All four form factors are to be determined experimentally.

The calculations in this paper can be applied to muon-proton scattering. Perhaps the additional form factors will explain the shrinking proton problem.

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References

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