KLEIN-GORDON EQUATION (A REVIEW)

The relativistic equation relating a particles energy E, three momentum $\mathbf{p}$, and mass m is

$$E^2 = (\mathbf{p})^2 + m^2. \tag{1}$$

Note that $\hbar$ and c have been set equal to 1. The relativistic wave equation for a free particle of spin zero is found by making the replacements $E \to i\partial/\partial t$ and $\mathbf{p} \to -i\nabla$ in Eq. (1). and applying the resulting operator to the free particle wave function $\phi(x)$. The resulting equation

$$\left(\frac{\partial^2\phi(x)}{\partial t^2} - \nabla_x^2\phi(x)\right) + m^2\phi(x) = \frac{\partial}{\partial x^\mu}\frac{\partial\phi(x)}{\partial x_\mu} + m^2\phi(x) = 0 \tag{2}$$

is called the Klein-Gordon equation for a free particle. Note that $\mu = 0, 1, 2, 3$, and $x = (x^0 = t, x^1, x^2, x^3)$

The solutions to Eq. (2) are

$$\phi(x) = N \exp\left(\mp ip \cdot x\right) \tag{3}$$

where N is a normalization constant. An eigen-value equation for the energy takes the form

$$E_{op}[N \exp\left(-ip \cdot x\right)] = i\frac{\partial(N \exp\left(-ip \cdot x\right))}{\partial t} = EN \exp\left(-ip \cdot x\right), \tag{4}$$

Date: September 20, 2014.

so $N\exp(-ip\cdot x)$ is a positve energy solution. It can similarly be shown that $N\exp(+ip\cdot x)$ is the negative energy solution.

The result of taking the complex conjugate of Eq. (2)is

$$\Big(\frac{\partial^2\phi^*(x)}{\partial t^2}-\nabla_x^2\phi^*(x)\Big)+m^2\phi^*(x)=\frac{\partial}{\partial x^\mu}\frac{\partial\phi^*(x)}{\partial x_\mu}+m^2\phi^*(x)=0\,. \quad (5)$$

Multiply Eq. (2) by ϕ^*, multiply Eq. (5) by ϕ, subtract the two resulting equations, and find

$$\phi^*(x)\partial_\mu\partial^\mu\phi(x)-\phi(x)\partial_\mu\partial^\mu\phi^*(x)=\partial_\mu(\phi^*(x)\partial^\mu\phi(x)-\phi(x)\partial^\mu\phi^*(x))=0\,. \quad (6)$$

Thus the solution to the free particle Klein-Gordon equation satisfies an equation of continuity $\partial_\mu j^\mu=0$ where the probability current density is identified as $j^\mu=i(\phi^*(x)\partial^\mu\phi(x)-\phi(x)\partial^\mu\phi^*(x))=(\rho,\mathbf{j})$. Thus the probability density is $\rho=j^0=i(\phi^*(x)\partial^0\phi(x)-\phi(x)\partial^0\phi^*(x))$.

Choose the normalization constant N so that there is unit probability that a positive energy Klein-Gordon particle is in a large box of volume V, i.e.,

$$\int\rho\,d^3x=\int i(\phi^*(x)\partial_0\phi(x)-\phi(x)\partial_0\phi^*(x))\,d^3x= \int 2ENN^*\,d^3x=2E|N|^2V=1. \quad (7)$$

So $N=1/\sqrt{2EV}$.

In addition, impose periodic boundary conditions on the wave function. Center the box at the origin. Then for a box side of length L_1

in say the x^1 direction, the wave function is required to be equal at $x^1 = \pm L_1/2$, i.e., $\exp(-ip_1 L_1/2) = \exp(+ip_1 L_1/2)$, or $\sin(p_1 L_1/2) = 0$. Thus $p_1 = 2\pi n_1/L_1$ where n_1 is an integer. The other components of $\mathbf{p}$ are $p_2 = 2\pi n_2/L_2$, and $p_3 = 2\pi n_3/L_3$.

For $\phi_f(x) = \exp(-ip_f \cdot x)/\sqrt{2E_f V}$, and $\phi_i(x) = \exp(-ip_i \cdot x)/\sqrt{2E_i V}$ when $\mathbf{p}_f \neq \mathbf{p}_i$,

$$\int i(\phi_f^*(x)\partial_0\phi_i(x) - \phi_i(x)\partial_0\phi_f^*(x))\, d^3x = \int \frac{\exp(i(p_f - p_i)\cdot x)}{2V\sqrt{E_f E_i}}[i(-iE_i) - i(iE_f)]d^3x = 0. \quad (8)$$

Eq. (8) follows since $\int \exp(-i(\mathbf{p}_f - \mathbf{p}_i)\cdot \mathbf{x})d^3x = 0$ when $\mathbf{p}_f \neq \mathbf{p}_i$. Thus with periodic boundary conditions, wave functions with different $\mathbf{p}$ are orthogonal. Eq. (7) and Eq. (8) can be combined into

$$\int i(\phi_f^*(x)\partial_0\phi_i(x) - \phi_i(x)\partial_0\phi_f^*(x))\, d^3x = \delta^3_{\mathbf{p}_f,\mathbf{p}_i} \quad (9)$$

where $\delta^3_{\mathbf{p}_f,\mathbf{p}_i} = 1$ when $\mathbf{p}_f = \mathbf{p}_i$, and $\delta^3_{\mathbf{p}_f,\mathbf{p}_i} = 0$ when $\mathbf{p}_f \neq \mathbf{p}_i$. For the negative energy solution

$$\int i(\phi_f^*(x)\partial_0\phi_i(x) - \phi_i(x)\partial_0\phi_f^*(x))\, d^3x = -\delta^3_{\mathbf{p}_f,\mathbf{p}_i}. \quad (10)$$

Let the volume $V \to \infty$, then the allowed momentum values become continuous instead of discrete. Normalize the wave functions in this case by requiring

$$\int i(\phi_f^*(x)\partial_0\phi_i(x) - \phi_i(x)\partial_0\phi_f^*(x))\, d^3x = \pm\delta^3(\mathbf{p}_f - \mathbf{p}_i) \quad (11)$$

where the plus sign on the right is for positive energy solutions, and the negative sign applies to the negative energy solutions. Substitute the wave functions into Eq. (11), and find

$$N^2(E_f + E_i)\exp[i(p_f^0 - p_i^0)x^0](2\pi)^3\delta^3(\mathbf{p}_f - \mathbf{p}_i) = \delta^3(\mathbf{p}_f - \mathbf{p}_i)\,. \quad (12)$$

or $N = 1/\sqrt{(2\pi)^3\, 2E_i}\,.$

To find the Klein-Gordon equation for a spin zero particle interacting with an electro-magnetic field, replace ∂_μ by $\partial_\mu + ieA_\mu$ where e is the charge of the particle, and A_μ is the is the four dimensional vector potential. The Klein-Gordon equation becomes

$$\frac{\partial}{\partial x^\mu}\frac{\partial\psi(x)}{\partial x_\mu} + ieA^\mu\frac{\partial\psi(x)}{\partial x^\mu} + ie\frac{\partial(A^\mu\psi(x))}{\partial x^\mu} - e^2A^\mu A_\mu\psi(x) + m^2\psi(x) = 0\,. \quad (13)$$

Introduce

$$V = +ieA_\mu\frac{\partial}{\partial x_\mu} + ie\frac{\partial A^\mu}{\partial x^\mu} - q^2A^\mu A_\mu\,. \quad (14)$$

Then the Klein-Gordon equation is written

$$\frac{\partial}{\partial x^\mu}\frac{\partial\psi(x)}{\partial x_\mu} + m^2\psi(x) = -V\psi(x)\,. \quad (15)$$