KLEIN-GORDON PROPAGATOR (A REVIEW)

Define the Klein-Gordon propagator $\Delta_F(x-y)$ by

$$\frac{\partial^2 \Delta_F(x-y)}{\partial (x^0)^2} - \nabla_x^2 \Delta_F(x-y) + m^2 \Delta_F(x-y) = -\delta^4(x-y). \quad (1)$$

Write $\Delta_F(x-y)$ as a Fourier transform, i.e.,

$$\Delta_F(x-y) = \int \frac{d^4k}{(2\pi)^4} \Delta_F(k) \exp\left(-ik\cdot(x-y)\right) \quad (2)$$

where the four-dimensional momentum vector $k = (k^0, k^1, k^2, k^3)$. Substitute Eq. (2) into Eq. (1), and find

$$\int \frac{d^4k}{(2\pi)^4} \Delta_F(k)(-k^2+m^2) \exp\left(-ik\cdot(x-y)\right) = \\ -\delta^4(x-y) = -\int \frac{d^4k}{(2\pi)^4} \exp\left(-ik\cdot(x-y)\right) \quad (3)$$

So

$$\Delta_F(k)(-k^2+m^2) = -1. \quad (4)$$

A solution to Eq. (4) is $\Delta_F(k) = 1/(k^2-m^2)$, but this is not the most general solution. The general solution is

$$\Delta_F(k) = 1/[(k^0)^2 - \mathbf{k}^2 - m^2] + C_1\delta(k^0 - \sqrt{\mathbf{k}^2+m^2}) + C_2\delta(k^0 + \sqrt{\mathbf{k}^2+m^2}) \quad (5)$$

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This can be verified by multiplying Eq. (5) by $[(k^0)^2-\mathbf{k}^2-m^2]$, and using $(k^0-\sqrt{\mathbf{k}^2+m^2})\delta(k^0-\sqrt{\mathbf{k}^2+m^2})=0$ and $(k^0+\sqrt{\mathbf{k}^2+m^2})\delta(k^0+\sqrt{\mathbf{k}^2+m^2})=0$. Eq. (4) is then recovered.

Use partial fractions to show

$$\frac{1}{(k^0)^2-\mathbf{k}^2-m^2}=\frac{1}{2\sqrt{\mathbf{k}^2+m^2)}}\Big(\frac{1}{k^0-\sqrt{\mathbf{k}^2+m^2}}-\frac{1}{k^0+\sqrt{\mathbf{k}^2+m^2}}\Big), \tag{6}$$

so

$$\begin{aligned}\Delta_F(k)=&\frac{1}{2\sqrt{\mathbf{k}^2+m^2)}}\Big(\frac{1}{k^0-\sqrt{\mathbf{k}^2+m^2}}\Big)+C_1\delta(k^0-\sqrt{\mathbf{k}^2+m^2})+\\&\frac{-1}{2\sqrt{\mathbf{k}^2+m^2)}}\Big(\frac{1}{k^0+\sqrt{\mathbf{k}^2+m^2}}\Big)+C_2\delta(k^0+\sqrt{\mathbf{k}^2+m^2}\end{aligned} \tag{7}$$

The integral I below is part of the integral in Eq. (3) and will be evaluated using reasonable physical constraints, which in turn will determine the constants C_1 and C_2:

$$I=\int_{-\infty}^{+\infty}dk^0\,\Delta_F(k)\,\exp\left(-ik^0(x^0-y^0)\right)=I_1+I_2 \tag{8}$$

where

$$I_1=\int_{-\infty}^{+\infty}dk^0\,\frac{1}{2\sqrt{\mathbf{k}^2+m^2)}}\Big(\frac{1}{k^0-\sqrt{\mathbf{k}^2+m^2}}-\frac{1}{k^0+\sqrt{\mathbf{k}^2+m^2}}\Big)\,\exp\left(-ik^0(x^0-y^0)\right), \tag{9}$$

and

$$I_2=\int_{-\infty}^{+\infty}dk^0\,[C_1\delta(k^0-\sqrt{\mathbf{k}^2+m^2})+C_2\delta(k^0+\sqrt{\mathbf{k}^2+m^2}]\,\exp\left(-ik^0(x^0-y^0)\right). \tag{10}$$

To evaluate I_1 using contour integration, introduce the closed contour below where ϵ is a small constant and M is a large real value of k^0, which eventually $\to \infty$:

(1) from $-M$ to $-\sqrt{\mathbf{k}^2 + m^2} - \epsilon$ along the real k^0 axis

(2) from $-\sqrt{\mathbf{k}^2 + m^2} - \epsilon$ to $-\sqrt{\mathbf{k}^2 + m^2} + \epsilon$ along a semi-circle of radius ϵ, which is centered at $-\sqrt{\mathbf{k}^2 + m^2}$, and is below the k^0 axis.

(3) from $-\sqrt{\mathbf{k}^2 + m^2} + \epsilon$ to $+\sqrt{\mathbf{k}^2 + m^2} - \epsilon$ along the k^0 axis

(4) from $+\sqrt{\mathbf{k}^2 + m^2} - \epsilon$ to $+\sqrt{\mathbf{k}^2 + m^2} + \epsilon$ along a semi-circle of radius ϵ, which is centered at $+\sqrt{\mathbf{k}^2 + m^2}$, and is below the k^0 axis.

(5) from $+\sqrt{\mathbf{k}^2 + m^2} + \epsilon$ to $+M$ along the k^0 axis

(6) $+M$ to $-M$ along a semi-circle S_M of radius M, which is centered at the origin of the k^0 axis and is below the axis.

This contour will be referred to as contour C1. There are no poles inside the contour, so the integral around the contour is zero. When $x^0 - y^0 > 0$ and $M \to \infty$, the integral vanishes on S_M. I_1 is identified as a Cauchy principle value. So

$$I_1 = -\pi i \mathrm{Res}(k^0 = +\sqrt{\mathbf{k}^2 + m^2}) - \pi i \mathrm{Res}(k^0 = -\sqrt{\mathbf{k}^2 + m^2}) =$$
$$\frac{-\pi i}{2\sqrt{\mathbf{k}^2 + m^2}} \Big(\exp\left(-i\sqrt{\mathbf{k}^2 + m^2}(x^0 - y^0)\right) - \exp\left(+i\sqrt{\mathbf{k}^2 + m^2}(x^0 - y^0)\right) \Big) H(x^0 - y^0) \tag{11}$$

where H is the unit step function defined by $H(x^0 - y^0) = 1$ if $x^0 - y^0 > 0$ and $H(x^0 - y^0) = 0$ if $x^0 - y^0 < 0$.

Next, evaluate I_1 when $x^0 - y^0 < 0$ by introducing the closed contour below:

(1) from $-M$ to $-\sqrt{\mathbf{k}^2 + m^2} - \epsilon$ along the k^0 axis

(2) from $-\sqrt{\mathbf{k}^2 + m^2} - \epsilon$ to $-\sqrt{\mathbf{k}^2 + m^2} + \epsilon$ along a semi-circle of radius ϵ, which is centered at $-\sqrt{\mathbf{k}^2 + m^2}$, and is above the k^0 axis.

(3) from $-\sqrt{\mathbf{k}^2 + m^2} + \epsilon$ to $+\sqrt{\mathbf{k}^2 + m^2} - \epsilon$ along the k^0 axis

(4) from $+\sqrt{\mathbf{k}^2 + m^2} - \epsilon$ to $+\sqrt{\mathbf{k}^2 + m^2} + \epsilon$ along a semi-circle of radius ϵ, which is centered at $+\sqrt{\mathbf{k}^2 + m^2}$, and is above the k^0 axis.

(5) from $+\sqrt{\mathbf{k}^2 + m^2} + \epsilon$ to $+M$ along the q^0 axis

(6) $+M$ to $-M$ along a semi-circle S_M of radius M, which is centered at the origin of the k^0 axis and is above the k^0 axis.

This contour will be referred to as contour C2. There are no poles inside the contour, so the integral around the contour is zero. When

$x^0 - y^0 < 0$ and $M \to \infty$, the integral vanishes on S_M. We identify I_1 as a Cauchy principle value, so

$$I_1 = +\pi i \mathrm{Res}(k^0 = +\sqrt{\mathbf{k}^2 + m^2}) + \pi i \mathrm{Res}(k^0 = -\sqrt{\mathbf{k}^2 + m^2}) =$$
$$\frac{+\pi i}{2\sqrt{\mathbf{k}^2 + m^2}} \Big(\exp\left(-i\sqrt{\mathbf{k}^2 + m^2}(x^0 - y^0)\right) - \exp\left(+i\sqrt{\mathbf{k}^2 + m^2}(x^0 - y^0)\right) \Big) H(y^0 - x^0). \tag{12}$$

Integrate I_2, and get

$$I_2 = C_1 \exp\left(-i\sqrt{\mathbf{k}^2 + m^2}(x^0 - y^0)\right) + C_2 \exp\left(+i\sqrt{\mathbf{k}^2 + m^2}(x^0 - y^0)\right). \tag{13}$$

Multiply I_2 by $H(x^0 - y^0) + H(y^0 - x^0) = 1$. Then Eq. (8) can be written

$$\begin{aligned} I = &\left(\frac{-\pi i}{2\sqrt{\mathbf{k}^2 + m^2}} + C_1 \right) \exp\left(-i\sqrt{\mathbf{k}^2 + m^2}(x^0 - y^0)\right) H(x^0 - y^0) + \\ &\left(\frac{+\pi i}{2\sqrt{\mathbf{k}^2 + m^2}} + C_2 \right) \exp\left(+i\sqrt{\mathbf{k}^2 + m^2}(x^0 - y^0)\right) H(x^0 - y^0) + \\ &\left(\frac{+\pi i}{2\sqrt{\mathbf{k}^2 + m^2}} + C_1 \right) \exp\left(-i\sqrt{\mathbf{k}^2 + m^2}(x^0 - y^0)\right) H(y^0 - x^0) + \\ &\left(\frac{-\pi i}{2\sqrt{\mathbf{k}^2 + m^2}} + C_2 \right) \exp\left(+i\sqrt{\mathbf{k}^2 + m^2}(x^0 - y^0)\right) H(y^0 - x^0) \end{aligned} \tag{14}$$

The second term represents a Klein-Gordon particle of negative energy traveling forward in time from y to x. This term is rejected by Feynman as being unphysical, so eliminate the second term by setting C_2 equal to $-\pi i/(2\sqrt{\mathbf{k}^2 + m^2})$. The third term represents a particle of positive energy traveling backward in time. This is unphysical, so eliminate this term by setting C_1 equal to $-\pi i/(2\sqrt{\mathbf{k}^2 + m^2})$. Put these results into Eq. (7), and find

$$\Delta_F(k) = \frac{1}{2\sqrt{\mathbf{k}^2+m^2})}\Big(\frac{1}{k^0-\sqrt{\mathbf{k}^2+m^2}} - \pi i\delta(k^0-\sqrt{\mathbf{k}^2+m^2})\Big)+ \frac{-1}{2\sqrt{\mathbf{k}^2+m^2})}\Big(\frac{1}{k^0+\sqrt{\mathbf{k}^2+m^2}} + \pi i\delta(k^0+\sqrt{\mathbf{k}^2+m^2}\Big). \quad (15)$$

The theory of generalized functions proves that $1/(x \pm i\epsilon) = 1/(x) \mp \pi i\delta(x)$.

Thus

$$\Delta_F(k) = \frac{1}{2\sqrt{\mathbf{k}^2+m^2})}\Big(\frac{1}{k^0-\sqrt{\mathbf{k}^2+m^2}+i\epsilon} - \frac{1}{k^0+\sqrt{\mathbf{k}^2+m^2}-i\epsilon}\Big). \quad (16)$$

An alternative expression for $\Delta_F(k)$ is

$$\Delta_F(k) = \frac{1}{(k^0)^2-\mathbf{k}^2-m^2+i\epsilon}. \quad (17)$$