KLEIN-GORDON SCATTERING AMPLITUDE
(A REVIEW)

The Klein-Gordon equation for a free particle is

$$\partial_\mu \partial^\mu \phi(x) + m^2 \phi(x) = 0. \tag{1}$$

The positive energy solution is $\phi(x) = N \exp(-ip \cdot x)$ where the normalization constant is chosen to be $N = 1/\sqrt{2EV}$ for a particle in a large box of volume V. The three-momentum $\mathbf{p}$ is determined by periodic boundary conditions, and the three-momentum values are discrete.

Let $V \to \infty$. Then the three-momentum values are continuous, and $N = 1/\sqrt{(2\pi)^3\, 2E}$. Recall or verify that

$$\int i\big(\phi_f^*(x)\partial_0\phi_n(x) - \phi_n(x)\partial_0\phi_f^*(x)\big)\, d^3x = \delta^3(\mathbf{p}_f - \mathbf{p}_n) \tag{2}$$

.

The Klein-Gordon propagator can be written as the Fourier transform

$$\Delta_F(x-y) = \int \frac{d^4k}{(2\pi)^4} \Delta_F(k) \exp\left(-ik \cdot (x-y)\right) \tag{3}$$

Date: October 19, 2014.

where

$$\Delta_F(k) = \frac{1}{(k^0)^2 - \mathbf{k}^2 - m^2 + i\epsilon} =$$
$$\frac{1}{2\sqrt{\mathbf{k}^2 + m^2)}}\Big(\frac{1}{k^0 - \sqrt{\mathbf{k}^2 + m^2} + i\epsilon} - \frac{1}{k^0 + \sqrt{\mathbf{k}^2 + m^2} - i\epsilon}\Big). \quad (4)$$

The Klein-Gordon equation for a particle in an electro-magnetic field is

$$\frac{\partial}{\partial x^\mu}\frac{\partial\psi(x)}{\partial x_\mu} + m^2\psi(x) = -\hat{V}(x)\psi(x) \quad (5)$$

where

$$\hat{V} = +ieA_\mu\frac{\partial}{\partial x_\mu} + ie\frac{\partial A^\mu}{\partial x^\mu} - e^2A^\mu A_\mu\,. \quad (6)$$

A^μ is the four-dimensional potential, and e is the charge of the particle. The solution to Eq. (5) is

$$\psi(x) = \phi(x) + \int \Delta_F(x-y)\hat{V}(y)\psi(y)d^4y\,. \quad (7)$$

To verify that Eq. (7) is a solution to Eq. (5), apply $(\partial_\mu\partial^\mu + m^2)$ to $\psi(x)$, and find

$$(\partial_\mu\partial^\mu{+}m^2)\psi(x) = (\partial_\mu\partial^\mu{+}m^2)\phi(x){+}\int(\partial_\mu\partial^\mu{+}m^2)\Delta_F(x{-}y)\hat{V}(y)\psi(y)d^4y$$
$$= \int -\delta^4(x-y)\hat{V}(y)\psi(y)d^4y = -\hat{V}(x)\psi(x), \quad (8)$$

which is just Eq. (5).

Consider the scattering of a positive energy Klein-Gordon particle. A wave of definite three-momentum $\mathbf{p}_i$ at $x^0 \to -\infty$ is sent into a region of space where there is an electro-magnetic potential A^μ. The integral

$$I = \lim_{x^0\to\infty} \int d^3x [\phi^*(x) i\partial_0 \psi(x) - \psi(x) i\partial_0 \phi^*(x)] \tag{9}$$

will be evaluated using two different expressions for $\psi(x)$. First, write $\psi(x) = \sum_n a_n \phi_n(x)$ as $x^0 \to \infty$ where $\sum_n$ is the sum over all possible plane wave states (actually an integral), and a_n is the probability amplitude that a particle emerges after scattering as $x^0 \to \infty$ in the plane wave state with three-momentum $\mathbf{p}_n$. Then

$$I = \lim_{x^0\to\infty} \int d^3x \sum_n a_n i\big(\phi_f^*(x)\partial_0\phi_n(x) - \phi_n(x)\partial_0\phi_f^*(x)\big) d^3x = a_f\,. \tag{10}$$

Next, evaluate Eq. (9) by using Eq. (7) for $\psi(x)$. Then

$$\begin{aligned} I = &\lim_{x^0\to\infty} \int d^3x \Big[\phi_f^*(x) i\partial_0\phi_i(x) - \phi_i(x) i\partial_0\phi_f(x) + \\ &\lim_{x^0\to\infty} \int d^3x \frac{\exp{(ip_f\cdot x)}}{\sqrt{2E_f V}} \frac{(k^0 + p_f^0)}{2\sqrt{\mathbf{k}^2 + m^2}} \exp{(-ik\cdot(x-y))} \frac{d^4k}{(2\pi)^4} \\ &\Big(\frac{1}{k^0 - \sqrt{\mathbf{k}^2+m^2} + i\epsilon} - \frac{1}{k^0 + \sqrt{\mathbf{k}^2+m^2} - i\epsilon}\Big) \hat{V}(y)\psi(y) d^4y. \end{aligned} \tag{11}$$

The first integral is Eq. (2). To integrate the second integral, introduce

$$\begin{aligned} I_S = \int \frac{dk^0}{2\pi} \frac{(k^0 + p_f^0)}{2\sqrt{\mathbf{k}^2+m^2}} \exp{[-ik^0(x^0 - y^0)]} \\ \Big(\frac{1}{k^0 - \sqrt{\mathbf{k}^2+m^2} + i\epsilon} - \frac{1}{k^0 + \sqrt{\mathbf{k}^2+m^2} - i\epsilon}\Big). \end{aligned} \tag{12}$$

Use contour integration, and find

$$\begin{aligned} I_S = \lim_{x^0\to\infty} -i \int \Big[&\frac{(\sqrt{\mathbf{k}^2+m^2} + p_f^0)}{2\sqrt{\mathbf{k}^2+m^2}} \exp{(-i\sqrt{\mathbf{k}^2+m^2}(x^0-y^0))} H(x^0 - y^0) \\ + &\frac{(-\sqrt{\mathbf{k}^2+m^2} + p_f^0)}{2\sqrt{\mathbf{k}^2+m^2}} \exp{(+i\sqrt{\mathbf{k}^2+m^2}(x^0-y^0))} H(y^0 - x^0)\Big]. \end{aligned} \tag{13}$$

Note that as $x^0 \to \infty$, $H(x^0 - y^0) = 1$, and $H(y^0 - x^0) = 0$. So

$$I = \delta^3(\mathbf{p}_f - \mathbf{p}_n) - i \int d^3x \, \frac{\exp{(ip_f \cdot x)}}{\sqrt{2E_f V}} I_S \exp{(+i\mathbf{k} \cdot (\mathbf{x} - \mathbf{y}))} \frac{d^3k}{(2\pi)^3} \hat{V}(y)\psi(y) d^4y. \tag{14}$$

Note that I contains the integral

$$I_{S1} = \int d^3x \, \exp{(+i(\mathbf{k} - \mathbf{p}_f) \cdot \mathbf{x})} = (2\pi)^3 \delta^3(\mathbf{k} - \mathbf{p}_f). \tag{15}$$

So

$$I = \delta^3(\mathbf{p}_f - \mathbf{p}_n) - i \int \exp{[i(p_f^0 - \sqrt{\mathbf{k}^2 + m^2}\,)x^0)]} \frac{(\sqrt{\mathbf{k}^2 + m^2} + p_f^0)}{2\sqrt{\mathbf{k}^2 + m^2}} (2\pi)^3 \delta^3(\mathbf{k} - \mathbf{p}_f) \frac{d^3k}{(2\pi)^3}$$
$$\frac{\exp{[i\sqrt{\mathbf{k}^2 + m^2}\, y^0 - i\mathbf{k} \cdot \mathbf{y}]}}{\sqrt{2E_f V}} \hat{V}(y)\psi(y) d^4y =$$
$$\delta^3(\mathbf{p}_f - \mathbf{p}_n) - i \int \exp{[i(p_f^0 - \sqrt{\mathbf{p}_f^2 + m^2}\,)x^0)]} \frac{(\sqrt{\mathbf{p}_f^2 + m^2} + p_f^0)}{2\sqrt{\mathbf{p}_f^2 + m^2}}$$
$$\frac{\exp{[i\sqrt{\mathbf{p}_f^2 + m^2}\, y^0 - i\mathbf{k} \cdot \mathbf{y}]}}{\sqrt{2E_f V}} \hat{V}(y)\psi(y) d^4y. \tag{16}$$

Use $p_f^0 = \sqrt{\mathbf{p}_f^2 + m^2}$, and find

$$I = \delta^3(\mathbf{p}_f - \mathbf{p}_i) - i \int \phi_f^*(y) \hat{V}(y) \psi(y) d^4y. \tag{17}$$

Thus $I = a_f = \delta^3(\mathbf{p}_f - \mathbf{p}_i) + \int \phi_f^*(y) \hat{V}(y) \psi(y) d^4y$. To agree with a common notation,[1] replace a_f by S_{fi}, and write

$$S_{fi} = \delta^3(\mathbf{p}_f - \mathbf{p}_i) - i \int \phi_f^*(y) \hat{V}(y) \psi(y) d^4y\,. \tag{18}$$

References

[1] J.D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill, New York, 1964), pp. 184-191.