COULOMB SCATTERING OF A PI MESON

Abstract

The pion charge is considered to be distributed or extended in space. The differential of the pion charge is set equal to a function of pion charge coordinates multiplied by a four-dimensional differential volume element. The four-dimensional integral of this function is required to equal the pion charge in all Lorentz frames. The scattering amplitude for such a pion in a Coulomb field is calculated. The result is that the scattering amplitude for the extended pion is the product of the scattering amplitude for a point pion times a pion form factor.

I. SCATTERING OF A POINT PI MESON BY A COULOMB POTENTIAL

The π^+ will be taken as the positive energy solution to the Klein-Gordon equation, so $e > 0$. The scattering amplitude is given by

$$S^+_{fi} = \delta^3(\mathbf{p}_f - \mathbf{p}_i) - i \int \phi^*_f(y)\hat{V}(y)\psi(y)d^4y \,.^1 \tag{1}$$

where $\phi_f(x)$ is a plane wave solution to the Klein-Gordon equation with normalization constant $N_f = 1/\sqrt{(2\pi)^3 2E_f}$. The scattered wave

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is

$$\psi(x) = \phi_i(x) + \int \Delta_F(x-y)\hat{V}(y)\psi(y)d^4y \qquad (2)$$

where $\Delta_F(x-y)$ is the Klein-Gordon propagator, and

$$\hat{V}(y) = +ieA^\mu(y)\frac{\partial}{\partial y^\mu} + ie\frac{\partial A^\mu(y)}{\partial y^\mu} - e^2A^\mu(y)A_\mu(y). \qquad (3)$$

The calculation will follow reference (1). Assume that the effect of the four-dimensional potential A^μ is small, and that $A^\mu A_\mu$ in $\hat{V}$ can be neglected. In addition, approximate $\psi(x)$ by $\phi(x)$ in Eq. (1). Then for $\mathbf{p}_f \neq \mathbf{p}_i$,

$$S^+_{fi} = \int \phi^*_f(y)\Big(ieA^\mu(y)\frac{\partial\phi_i(y)}{\partial y^\mu} + ie\frac{\partial(A^\mu(y)\phi_i(y))}{\partial y^\mu}\Big)d^4y\,. \qquad (4)$$

Use integration by parts to show

$$\int \phi^*_f(y)\frac{\partial(A^\mu(y)\phi_i(y))}{\partial y^\mu}d^4y = -\int \frac{\partial\phi^*_f(y)}{\partial y^\mu}(A^\mu(y)\phi_i(y))d^4y\,, \qquad (5)$$

which follows since all the three-dimensional integrals are zero. Then

$$S^+_{fi} = -i\int ie\Big(\phi^*_f(y)A^\mu(y)\frac{\partial\phi_i(y)}{\partial y^\mu} - \frac{\partial\phi^*_f(y)}{\partial y^\mu}(A^\mu(y)\phi_i(y))\Big)d^4y =$$

$$-i\int ie(-i)(p_i+p_f)_\mu\phi^*_f(y)A^\mu(y)\phi_i(y)d^4y\,. \qquad (6)$$

For a stationary nucleus with Z protons, take $A^0(y) = +Ze/(4\pi|\mathbf{y}|)$, and $\mathbf{A} = 0$. Then

$$S^*_{fi} = -ie\int (p_i + p_f)_0 \frac{Ze}{4\pi|\mathbf{y}|}\frac{\exp{(i(p_f - p_i)\cdot y)}}{(2\pi)^3\sqrt{2E_i 2E_f}}d^4y =$$

$$\frac{-iZe^2 2\pi\delta(p_f^0 - p_i^0)(p_f^0 + p_i^0)}{(2\pi)^3\sqrt{2E_i 2E_f}}\int\frac{\exp{(i(\mathbf{p}_f - \mathbf{p}_i)\cdot \mathbf{y})}d^3y}{4\pi|\mathbf{y}|} =$$

$$\frac{-iZe^2 2\pi\delta(E_f - E_i)}{(2\pi)^3\sqrt{2E_i 2E_f}}\frac{(E_f + E_i)}{|\mathbf{p}_f - \mathbf{p}_i|^2} = \frac{-iZe^2 2\pi\delta(E_f - E_i)}{(2\pi)^3|\mathbf{p}_f - \mathbf{p}_i|^2}. \quad (7)$$

II. EXTENDED PION COORDINATES

In the rest frame of a pion charge distribution, let $x'^{\mu}_r = (x'^0_r, x'^1_r, x'^2_r, x'^3_r)$ denote a spacetime charge point. Let $x^\mu_r = (x^0_r, x^1_r, x^2_r, x^3_r)$ denote the center of the charge distribution. The charge distribution of the pion is assumed to have a well-defined center, which is identified as the argument of the wave function. The shape of the charge distribution depends on the motion of the charge, and is assumed to be unaffected by any interaction. Sometimes the superscript will be omitted, and we will write $x'_r = (x'^0_r, x'^1_r, x'^2_r, x'^3_r)$ and $x_r = (x^0_r, x^1_r, x^2_r, x^3_r)$. Introduce $\tilde{x}_r = x'_r - x_r$ or equivalently $\tilde{x}^\mu_r = x'^\mu_r - x^\mu_r$.

In a frame of reference in which the charge distribution moves with a constant speed β in the $+x^3$ direction, let $x'_m = (x'^0_m, x'^1_m, x'^2_m, x'^3_m)$ denote a spacetime charge point, and let $x_m = (x^0_m, x^1_m, x^2_m, x^3_m)$ denote the center of the charge distribution. Introduce $\tilde{x}_m = x'_m - x_m$. A

Lorentz transformation yields $\tilde{x}_r^1 = \tilde{x}_m^1$, $\tilde{x}_r^2 = \tilde{x}_m^2$, $\tilde{x}_r^3 = \gamma(\tilde{x}_m^3 - \beta\tilde{x}_m^0)$, and $\tilde{x}_r^0 = \gamma(\tilde{x}_m^0 - \beta\tilde{x}_m^3)$ where $\gamma = 1/\sqrt{1-\beta^2}$.

In the rest frame, the pion charge e is equal to $\int \rho_r(\tilde{x}_r)\delta(\tilde{x}_r^0)d^4\tilde{x}_r$ where $\rho_r(\tilde{x}_r)$ is the charge density in the rest frame.[2] So an element of charge in the rest frame is given by

$$de(\tilde{x}_r) = \rho_r(\tilde{x}_r)\delta(\tilde{x}_r^0)\, d^4\tilde{x}_r. \tag{8}$$

In the moving frame described above, the element of charge is

$$de(\tilde{x}_m) = \rho_r(\tilde{x}_m)\delta(\gamma(\tilde{x}_m^0 - \beta\tilde{x}_m^3)\, d^4\tilde{x}_m.^{\,3} \tag{9}$$

III. EXTENDED PI SCATTERING BY A COULOMB POTENTIAL

Take the π^+ meson to be initially moving in the $+y^3$ direction. This was previously called the m frame. The interaction takes place at the charge points, so in Eq. (6), replace e by the four-dimensional integral of $de(\tilde{y}_m)$. Also, $A^\mu(y) \to A^0(y')$. Then the scattering amplitude for the extended pion is

$$S_{FI}^* = -ie\int (p_i + p_f)_0 \frac{Ze}{4\pi|\mathbf{y}'|}\frac{\exp\left(i(p_f - p_i)\cdot y\right)}{(2\pi)^3\sqrt{2E_i 2E_f}}d^4y$$
$$\frac{\rho_r(\tilde{y}_m)}{e}\delta(\gamma(\tilde{y}_m^0 - \beta\tilde{y}_m^3)\, d^4\tilde{y}_m. \tag{10}$$

Change variables from y to y', and find $S^{+}_{FI} = I_S\, F(p_f - p_i)$ where

$$I_S = \frac{-iZe^2(E_i + E_f)}{(2\pi)^3\sqrt{2E_i 2E_f}} \int \exp\left(i(p^0_f - p^0_i)y'^0\right) dy'^0 \int \frac{\exp\left(i(\mathbf{p}_f - \mathbf{p}_i)\cdot \mathbf{y}'\right) d^3y'}{4\pi|\mathbf{y}'|}, \tag{11}$$

and the pion form factor $F(p_f - p_i)$ is

$$F(p_f - p_i) = \int \exp\left(-i(p_f - p_i)\cdot \tilde{y}\right) \frac{\rho_r(\tilde{y}_m)}{e} \delta(\gamma(\tilde{y}^0_m - \beta\tilde{y}^3_m)\, d^4\tilde{y}_m. \tag{12}$$

Integration yields

$$I_S = \frac{-iZe^2(E_i + E_f)}{(2\pi)^3\sqrt{2E_i 2E_f}} \frac{2\pi\delta(p^0_f - p^0_i)}{|\mathbf{p}_f - \mathbf{p}_i|^2} = S^{*}_{fi}. \tag{13}$$

Write $F(p_f - p_i)$ as

$$F(p_f - p_i) = \int \exp\left[+i(\tilde{y}^1_m p^1_f + \tilde{y}^2_m p^2_f + \tilde{y}^3_m(p^3_f - p^3_i) - \tilde{y}^0_m(p^0_f - p^0_i))\right] \frac{\rho_r(\tilde{y}_m)}{e} \delta(\gamma(\tilde{y}^0_m - \beta\tilde{y}^3_m)\, d^4\tilde{y}_m. \tag{14}$$

For the purpose of illustration, pick

$$\rho_r(\tilde{y}_m) = \frac{e\delta(\sqrt{(\tilde{y}^1_m)^2 + (\tilde{y}^2_m)^2 + \gamma^2(\tilde{y}^3_m - \beta\tilde{y}^0_m)^2} - a)}{4\pi a^2}, \tag{15}$$

and substitute Eq. (15) in Eq. (14). Then

$$F(p_f - p_i) = \int \exp\left[+i(\tilde{y}^1_m p^1_f + \tilde{y}^2_m p^2_f + \tilde{y}^3_m(p^3_f - p^3_i))\right] \frac{\delta(\sqrt{(\tilde{y}^1_m)^2 + (\tilde{y}^2_m)^2 + (\tilde{y}^3_m)^2/\gamma^2} - a)}{4\pi a^2} \frac{d^3\tilde{y}_m}{\gamma}. \tag{16}$$

Introduce $\hat{y}^1 = \tilde{y}^1_m$, $\hat{y}^2 = \tilde{y}^2_m$, and $\hat{y}^3 = \tilde{y}^1_m/\gamma$, and use $p^3_i = \beta p^0_i = \beta p^0_f$.

Then

$$F(p_f - p_i) = \int \exp\left[+i(\hat{y}^1 p^1_f + \hat{y}^2 p^2_f + \hat{y}^3 \gamma (p^3_f - \beta p^0_f))\right] \frac{\delta(|\hat{\mathbf{y}}| - a)}{4\pi a^2} d^3\hat{y}. \tag{17}$$

Introduce $|\hat{\mathbf{p}}_\mathbf{f}| = \sqrt{(p^1_f)^2 + (p^2_f)^2 + \gamma^2 (p^3_f - \beta p^0_f)^2}$, and get

$$F(p_f - p_i) = F(p_f) = j_0(|\hat{\mathbf{p}}_\mathbf{f}| a) \tag{18}$$

Putting these results together gives

$$S^+_{fi} = \frac{-iZe^2 2\pi \delta(p^0_f - p^0_i)}{(2\pi)^3 |\mathbf{p}_f - \mathbf{p}_i|^2} j_0(|\hat{\mathbf{p}}_\mathbf{f}| a). \tag{19}$$

ACKNOWLEDGMENTS

References

[1] J.D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill, New York, 1964), pp. 190-192.

[2] S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity* (John Wiley & Sons, Inc. New York, 1972), pp. 40-41.

[3] http://www.electronformfactor.com/Mott-Rutherford Scattering and Beyond