FINITE SELF MASS OF THE ELECTRON

ABSTRACT

The self-mass of an electron is found to be finite when the electron charge is considered to be distributed or extended in space. The differential of the electron charge is set equal to a function of electron charge coordinates multiplied by a four-dimensional differential volume element. The four-dimensional integral of this function is required to equal the electron charge in all Lorentz frames.

I. LOGARITHMIC DIVERGENCE OF THE SELF-MASS

The electron self-mass is associated with the following sequence of events: a point electron propagates from spacetime point y to spacetime point w where it emits a photon, the electron then propagates to spacetime point z where it absorbs the same photon, and finally the electron then propagates to spacetime point x. The propagator for this process, which is second order in the charge e,[1] is

$$S_{F2}(x-y) = \int S_F(x-z)(-ie\gamma^\mu)S_F(z-w)(-ie\gamma_\mu) S_F(w-y)D_F(z-w)d^4z\, d^4w \quad (1)$$

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where γ^μ and γ_μ ($\mu = 0, 1, 2, 3$) both represent the Dirac matrices. The electron propagators, $S_F(x-z), S_F(z-w)$, and $S_F(w-y)$, are written as Fourier integrals:

$$S_F(x-z) = \int \exp\left(-ip' \cdot (x-z)\right) S_F(p') \frac{d^4p'}{(2\pi)^4}\,, \tag{2}$$

$$S_F(z-w) = \int \exp\left(-ik \cdot (z-w)\right) S_F(k) \frac{d^4k}{(2\pi)^4}\,, \tag{3}$$

and

$$S_F(w-y) = \int \exp\left(-ip \cdot (w-y)\right) S_F(p) \frac{d^4p}{(2\pi)^4}\,, \tag{4}$$

where by choice

$$S_F(p') = \frac{i}{\not{p}' - m_0} = \frac{i(\not{p}' + m_0)}{p'^2 - m_0^2}, \tag{5}$$

$$S_F(p) = \frac{i}{\not{p} - m_0} = \frac{i(\not{p} + m_0)}{p^2 - m_0^2}, \tag{6}$$

and

$$S_F(k) = \frac{i(\not{k} + m_0)}{k^2 - m_0^2 + i\epsilon}. \tag{7}$$

Here p, p', and k are four-momentum vectors, $\not{p}' = \gamma^\mu p'_\mu$, $\not{p} = \gamma^\mu p_\mu$, $\not{k} = \gamma^\mu k_\mu$, and m_0 is called the bare electron mass. The observable or physical mass is given by $m = m_0 + \delta m$ where δm is the self-mass. The photon propagator is given by

$$D_F(z-w) = -i \int \frac{d^4q}{(2\pi)^4} \frac{\exp\left(-iq \cdot (z-w)\right)}{\left(q^2 - \mu^2 + i\epsilon\right)} \tag{8}$$

where q is a four-momentum vector, and μ is a small photon mass, which is inserted in the photon propagator to eliminate the infra-red divergence. Then Eq. (1) becomes

$$S_{F2}(x-y) = \int \frac{d^4p'}{(2\pi)^4} \exp\left(-ip'\cdot(x-z)\right) \frac{i(\not{p}'+m_0)}{(p'^2-m_0^2)}$$
$$\frac{(-ie\gamma^\mu)i(\not{k}+m_0)(-ie\gamma_\mu)}{k^2-m_0^2+i\epsilon} \exp\left(-ik\cdot(z-w)\right)\frac{d^4k}{(2\pi)^4}$$
$$(-i)\frac{d^4q}{(2\pi)^4}\frac{\exp\left(-iq\cdot(z-w)\right)}{q^2-\mu^2+i\epsilon} d^4z\, d^4w$$
$$\exp\left(-ip\cdot(w-y)\right)\frac{i(\not{p}+m_0)}{(p^2-m_0^2)}\frac{d^4p}{(2\pi)^4}. \quad (9)$$

Note that $\gamma^\mu(\not{k}+m_0)\gamma_\mu = -2\not{k}+4m_0$. Rearrange the terms to get

$$S_{F2}(x-y) = (-ie)^2 i(-i) \int \frac{d^4p'}{(2\pi)^4} \exp\left(-ip'\cdot x)\right) \frac{i(\not{p}'+m_0)}{(p'^2-m_0^2)}$$
$$\frac{(-2\not{k}+4m_0)}{k^2-m_0^2+i\epsilon}\frac{d^4k}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\frac{1}{q^2-\mu^2+i\epsilon}\, d^4z\, \exp\left(iz\cdot(-k-q+p')\right)$$
$$d^4w\, exp(iw\cdot(+k+q-p))\, exp(+ip\cdot y)\frac{i(\not{p}+m_0)}{(p^2-m_0^2)}\frac{d^4p}{(2\pi)^4}. \quad (10)$$

Note that

$$\int d^4z\, \exp\left(iz\cdot(-k-q+p')\right) d^4w\, exp(iw\cdot(+k+q-p)) =$$
$$(2\pi)^4\delta^4(-k-q+p')(2\pi)^4\delta(k+q-p) =$$
$$(2\pi)^4\delta^4(p'-p)\,(2\pi)^4\delta^4(k+q-p). \quad (11)$$

Substitute Eq. (11) into Eq. (10), and find

$$S_{F2}(x-y) = -e^2 \int \frac{d^4p'}{(2\pi)^4} \exp\left(-ip'\cdot x\right)) \frac{i(\not{p}'+m_0)}{(p'^2-m_0^2)} (2\pi)^4 \delta^4(p'-p)$$
$$\frac{(-2\not{k}+4m_0)}{k^2-m_0^2+i\epsilon} \frac{d^4k}{(2\pi)^4} \frac{d^4q}{(2\pi)^4} \frac{1}{q^2-\mu^2+i\epsilon}$$
$$(2\pi)^4 \delta^4(k+q-p) \exp\left(+ip\cdot y\right) \frac{i(\not{p}+m_0)}{(p^2-m_0^2)} \frac{d^4p}{(2\pi)^4}. \quad (12)$$

Perform the integration over the four components of p' and the four components of k, and find

$$S_{F2}(x-y) = -e^2 \int \exp\left(-ip\cdot x\right)) \frac{i(\not{p}+m_0)}{(p^2-m_0^2)} \frac{(-2\not{p}+2\not{q}+4m_0)}{(p-q)^2-m_0^2+i\epsilon}$$
$$\frac{d^4q}{(2\pi)^4} \frac{1}{q^2-\mu^2+i\epsilon} \exp\left(+ip\cdot y\right) \frac{i(\not{p}+m_0)}{(p^2-m_0^2)} \frac{d^4p}{(2\pi)^4}. \quad (13)$$

Write Eq. (13) in the form

$$S_{F2}(x-y) = \int \exp\left(-ip\cdot x\right)) \frac{i(\not{p}+m_0)}{(p^2-m_0^2)} (-i\Sigma_2(p))$$
$$\exp\left(+ip\cdot y\right) \frac{i(\not{p}+m_0)}{(p^2-m_0^2)} \frac{d^4p}{(2\pi)^4} \quad (14)$$

where $\Sigma_2(p)$ is given by

$$\Sigma_2(p) = -ie^2 \int \Big(\frac{-2\not{p}+2\not{q}+4m_0}{(p-q)^2-m_0^2+i\epsilon} \Big) \frac{1}{q^2-\mu^2+i\epsilon} \frac{d^4q}{(2\pi)^4}. \quad (15)$$

$\Sigma_2(\not p = m_0)$ is identified approximately as the self-mass.[1] Feynman's method of integration is to use

$$\frac{1}{AB} = \int_0^1 \frac{dt}{[B + (A - B)t]^2}. \tag{16}$$

Set $A = q^2 - 2q \cdot p + p^2 - m_0^2 + i\epsilon$, and set $B = q^2 - \mu^2 + i\epsilon$. Then

$$\Sigma_2(p) = -ie^2 \int_0^1 dt \int \frac{-2\not p + 2\not q + 4m_0}{[(q - pt)^2 - p^2t^2 + (p^2 - m_0^2)t - \mu^2(1-t) + i\epsilon]^2} \frac{d^4q}{(2\pi)^4}\,. \tag{17}$$

Change variables to $\ell = q - pt$, and get

$$\Sigma_2(p) = -ie^2 \int_0^1 dt \int \frac{-2\not p + 2\not \ell + 2\not p t + 4m_0}{[(\ell)^2 - p^2t^2 + (p^2 - m_0^2)t - \mu^2(1-t) + i\epsilon]^2} \frac{d^4\ell}{(2\pi)^4}\,. \tag{18}$$

The term linear in ℓ in the numerator integrates to zero. Note that if $\not p = m_0$, then $p^2 = m_0^2$, and

$$\Sigma_2(\not p = m_0) = -ie^2 \int_0^1 dt \int \frac{+2m_0(1+t)}{[(\ell^0)^2 - (\vec{\ell}^{\,2} + m_0^2t^2 + \mu^2(1-t)) + i\epsilon]^2} \frac{d^4\ell}{(2\pi)^4}\,. \tag{19}$$

Introduce $\Delta = +m_0^2t^2 + \mu^2(1-t)$. Then

$$\Sigma_2(\not p = m_0) = -ie^2 \int_0^1 dt \int \frac{2m_0(1+t)}{[(\ell^0) - \sqrt{(\vec{\ell}^{\,2} + \Delta} + i\epsilon]^2\,[(\ell^0) + \sqrt{(\vec{\ell}^{\,2} + \Delta} - i\epsilon]^2} \frac{d^4\ell}{(2\pi)^4}\,. \tag{20}$$

Introduce

$$I_S = \int \frac{1}{[(\ell^0) - \sqrt{(\vec{\ell}^{\,2} + \Delta} + i\epsilon]^2\,[(\ell^0) + \sqrt{(\vec{\ell}^{\,2} + \Delta} - i\epsilon]^2} \frac{d\ell^0}{(2\pi)}\,. \tag{21}$$

Notice the integrand has two double poles. Integrate from $-\infty$ to $+\infty$ along the real ℓ^0 axis. Complete the contour on the infinite semi-circle below the ℓ^0 axis. The residue of the pole at $\ell^0 = \sqrt{(\vec{\ell})^{\,2} + \Delta} - i\epsilon$ is given by

$$\frac{\partial}{\partial \ell^0}\Big(\frac{1}{\ell^0 + \sqrt{(\vec{\ell})^{\,2} + \Delta} + i\epsilon)}\Big)^2\Big|_{\ell^0 = \sqrt{(\vec{\ell})^{\,2}+\Delta}+i\epsilon} \quad = \quad \frac{-1}{4(\vec{\ell}^{\,2} + \Delta)^{3/2}}. \qquad (22)$$

Then

$$\Sigma_2(\not{p} = m_0) = -ie^2 \frac{(-2\pi i)}{(2\pi)} \int_0^1 dt \int \frac{-2m_0(1+t)}{4(\vec{\ell}^{\,2} + \Delta)^{3/2}} \frac{d^3\ell}{(2\pi)^3}. \qquad (23)$$

For large $|\vec{\ell}|$, $\Sigma_2(\not{p} = m_0) \to \int |\vec{\ell}|^2 d|\vec{\ell}|/|\vec{\ell}|^3 \to \log|\vec{\ell}|$ as $|\vec{\ell}| \to \infty$. Thus the self-mass diverges logarithmically.

II. EXTENDED ELECTRON COORDINATES

In the rest frame of an electron charge distribution, let $x_r'^{\mu} = (x_r'^{0}, x_r'^{1}, x_r'^{2}, x_r'^{3})$ denote a spacetime charge point. Let $x_r^{\mu} = (x_r^0, x_r^1, x_r^2, x_r^3)$ denote the center of the charge distribution. The charge distribution of the electron is assumed to have a well-defined center, which is identified as the argument of the wave function. The shape of the charge distribution depends on the motion of the charge, and is assumed to be unaffected by any interaction. Sometimes the superscript will be omitted, and we will write $x_r' = (x_r'^{0}, x_r'^{1}, x_r'^{2}, x_r'^{3})$ and $x_r = (x_r^0, x_r^1, x_r^2, x_r^3)$. Introduce $\tilde{x}_r = x_r' - x_r$ or equivalently $\tilde{x}_r^{\mu} = x_r'^{\mu} - x_r^{\mu}$.

In the rest frame, the electron charge e is equal to $\int \rho_r(\tilde{x}_r)\delta(\tilde{x}_r^0)d^4\tilde{x}_r$ where $\rho_r(\tilde{x}_r)$ is the charge density in the rest frame.[2,3] So an element of charge in the rest frame is given by

$$de(\tilde{x}_r) = \rho_r(\tilde{x}_r)\delta(\tilde{x}_r^0)\, d^4\tilde{x}_r. \tag{24}$$

III. FINITE SELF-MASS OF THE EXTENDED ELECTRON

To take into account electron size at emission and at absorption replace the first e by the four-dimensional integral of $de(\tilde{z})$ and replace the second e by the four-dimensional integral of $de(\tilde{w})$. In addition, replace $D_F(z-w)$ by $D_F(z'-w')$. Then Eq. (1) is replaced by

$$S'_{F2}(x-y) = \int S_F(x-z)(-ide(\tilde{z})\gamma^\mu)S_F(z-w)(-ide(\tilde{w})\gamma_\mu) S_F(w-y)D_F(z'-w')d^4z\, d^4w \tag{25}$$

Use $D_F(z'-w') = D_F(z-w)\exp(-iq\cdot\tilde{z})\exp(+iq\cdot\tilde{w})$ and find

$$S'_{F2}(x-y) = \int S_F(x-z)(-ie\gamma^\mu)S_F(z-w)(-ie\gamma_\mu)S_F(w-y) D_F(z-w)d^4z\, d^4w\, \exp(-iq\cdot\tilde{z})\frac{de(\tilde{z})}{e}\exp(+iq\cdot\tilde{w})\frac{de(\tilde{w})}{e} \tag{26}$$

It is convenient to continue the calculation in the frame in which the electron is initially at rest. The motion of the electron should not be changed by the emission of photons in all directions, or by the absorption of these photons, so it is assumed that the electron remains at rest. Then $de(\tilde{z})$ and $de(\tilde{w})$ have the form of Eq. (24). For the purpose of illustration, choose the electron charge to be uniformly distributed on a spherical shell of radius a, so

$$\rho_r(\tilde{w}_r) = e\frac{\delta(\sqrt{(\tilde{w}_r^1)^2 + (\tilde{w}_r^2)^2 + (\tilde{w}_r^3)^2} - a)}{4\pi a^2} = e\frac{\delta(|\tilde{\mathbf{w}}_r| - a)}{4\pi a^2} \qquad (27)$$

with a similar equation for $\rho_r(\tilde{z}_r)$. So $F_i(q)$, one electron form factor, is

$$F_i(q) = \int \exp\left(+iq \cdot \tilde{w}_r\right)\frac{de(\tilde{w}_r)}{e} =$$
$$\int \exp\left(-i\mathbf{q} \cdot \tilde{\mathbf{w}}_r\right)\frac{\delta(|\tilde{\mathbf{w}}_r| - a)}{4\pi a^2}\, d^3\tilde{w}_r = \frac{\sin\left(|\mathbf{q}|a\right)}{(|\mathbf{q}|a)} = j_0(|\mathbf{q}|a), \quad (28)$$

and the other electron form factor is

$$F_f(q) = \int \exp\left(-iq \cdot \tilde{z}_r\right)\frac{de(\tilde{z}_r)}{e} =$$
$$\int \exp\left(+i\mathbf{q} \cdot \tilde{\mathbf{z}}_r\right)\frac{\delta(|\tilde{\mathbf{z}}_r| - a)}{4\pi a^2}\, d^3\tilde{z}_r = \frac{\sin\left(|\mathbf{q}|a\right)}{(|\mathbf{q}|a)} = j_0(|\mathbf{q}|a) \quad (29)$$

where j_0 is the spherical Bessl function of order zero. Thus in the rest frame of the electron,

$$S'_{F2}(x-y) = \int S_F(x-z)(-ie\gamma^\mu)S_F(z-w)(-ie\gamma_\mu)S_F(w-y)$$
$$D_F(z-w)d^4z\, d^4w\;\; j_0^2(|\mathbf{q}|a) \quad (30)$$

Proceed as in section I, and find that Eq. (23) is replaced by

$$\Sigma_2'(\not p = m_0, \mathbf{p} = 0) = +ie^2 \frac{(-2\pi i)}{(2\pi)} \int_0^1 dt \int \frac{2m_0(1+t)}{4(\vec{\ell}^{\,2} + \Delta)^{3/2}} \frac{d^3\ell}{(2\pi)^3} j_0^2(|\vec{\ell}|a), \tag{31}$$

which is approximately the self-mass of the extended electron. Take the absolute value of the integrand, and observe that for large $|\vec{\ell}|$

$$\text{integral} \to \int \frac{|\vec{\ell}|^2 d|\vec{\ell}|}{(|\vec{\ell}|^2 + \Delta)^{3/2}} \frac{1}{|\vec{\ell}|^2} \to \int \frac{d|\vec{\ell}|}{|\vec{\ell}|^3}. \tag{32}$$

Thus the integral converges absolutely, so the self-mass for the chosen charge distribution is finite. Incidentally Eq. (31) can be integrated exactly in terms of modified Bessel functions. This is done in the appendix.

IV. AN ALTERNATIVE ELECTRON CHARGE DISTRIBUTION

Consider the electron charge uniformly distributed over a sphere of radius a in the rest frame. Then

$$\rho_r(\tilde{w}_r) = 3e \frac{H(a - \sqrt{(\tilde{w}_r^1)^2 + (\tilde{w}_r^2)^2 + (\tilde{w}_r^3)^2}\,)}{4\pi a^3} = 3e \frac{H(a - |\tilde{\mathbf{w}}_r|)}{4\pi a^3} \tag{33}$$

Then the electron form factor takes the form

$$F_i(q) = \int \exp(+iq \cdot \tilde{w}_r) \frac{de(\tilde{w}_r)}{e} = \int \exp(-i\mathbf{q} \cdot \tilde{\mathbf{w}}_r) \frac{3H(a - |\tilde{\mathbf{w}}_r|)}{4\pi a^3}\, d^3\tilde{w}_r =$$
$$\frac{3}{(|\mathbf{q}|a)^3}(sin(|\mathbf{q}|a) - (|\mathbf{q}|a)\cos(|\mathbf{q}|a)) = \frac{3}{(|\mathbf{q}|a)} j_1(|\mathbf{q}|a), \tag{34}$$

where j_1 is the spherical Bessel function of order one. Note that

$$F_f(q) = \int \exp(-iq \cdot \tilde{z}_r) \frac{de(\tilde{z}_r)}{e} = \frac{3}{(|\mathbf{q}|a)} j_1(|\mathbf{q}|a). \tag{35}$$

Take the absolute value of the integrand, and observe that for large $|\vec{\ell}|$

$$\text{integral} \to \int \frac{|\vec{\ell}|^2 d|\vec{\ell}|}{(|\vec{\ell}|^2 + \Delta)^{3/2}} \frac{1}{|\vec{\ell}|^4} \to \int \frac{d|\vec{\ell}|}{|\vec{\ell}|^5}. \tag{36}$$

Thus the integral converges absolutely, so the self-mass for this charge distribution is finite.

V. DISCUSSION

The calculations in the previous two sections suggest, but do not prove, that for a finite distribution of charge, the self-mass is finite.

APPENDIX

Eq. (31) can be written as

$$\Sigma_2'(\not{p} = m_0, \mathbf{p} = 0) = +ie^2 \frac{(-2\pi i)}{(2\pi)} \int_0^1 dt \int \frac{2m_0(1+t)}{4(\vec{\ell}^{\,2} + \Delta)^{3/2}} \frac{d^3\ell}{(2\pi)^3} \frac{\exp(2i|\vec{\ell}|a) + \exp(-2i|\vec{\ell}|a) - 2}{(2i|\vec{\ell}|a)^2}. \tag{37}$$

Note that $d^3\ell = |\vec{\ell}|^2 \, d\Omega \, d|\vec{\ell}|$, and the integral of $d\Omega$ is 4π. Also,

$$\int_0^\infty \frac{\exp(2i|\vec{\ell}|a) d|\vec{\ell}|}{(|\vec{\ell}|^2 + \Delta)^3} = \int_0^{-\infty} \frac{\exp(-2i|\vec{\ell}|a)(-d|\vec{\ell}|)}{(|\vec{\ell}|^2 + \Delta)^3} = \int_{-\infty}^0 \frac{\exp(-2i|\vec{\ell}|a) d|\vec{\ell}|}{(|\vec{\ell}|^2 + \Delta)^3}. \tag{38}$$

Then

$$\Sigma_2'(\not p = m_0, \mathbf{p} = 0) = -\frac{e^2 m_0}{8a^2}\frac{(8\pi)}{(2\pi)^3}\int_0^1 (1+t)dt(I_1 + I_2). \tag{39}$$

where

$$I_2 = -2\int_0^{+\infty} \frac{d|\vec{\ell}|}{(\vec{\ell}^{\,2} + \Delta)^{3/2}} = -2/\Delta, \tag{40}$$

and

$$I_1 = \int_{-\infty}^{+\infty} \frac{\exp\left(-2i|\vec{\ell}|a\right) d|\vec{\ell}|}{(\vec{\ell}^{\,2} + \Delta)^{3/2}}. \tag{41}$$

The substitutions $|\vec{\ell}| = \Delta^{1/2}|\vec{\ell}'|$, and $\zeta = 2\Delta^{1/2}a$ yield[4]

$$I_1 = \int_{-\infty}^{+\infty} \frac{\exp\left(-2i|\vec{\ell}|a\right) d|\vec{\ell}|}{(\vec{\ell}^{\,2} + \Delta)^{3/2}} = \int_{-\infty}^{+\infty} \frac{\exp\left(-i\zeta|\vec{\ell}'|\right) d|\vec{\ell}'|}{\Delta(\vec{\ell}'^{\,2} + 1)^{3/2}} = \frac{\sqrt{\pi}|\zeta|\, K_1(|\zeta|)}{\Delta(1/2)!2^0}. \tag{42}$$

where $(1/2)! = \sqrt{\pi}/2$,[5] and $K_1(\zeta)$ is the modified Bessel function of order 1. The argument of K_1 is small, so the first few terms of a Taylor series will be used to approximate K_1. Note that

$$K_1(x) = \frac{\pi\, i^2}{2} H_1^{(1)}(ix) = -\frac{\pi}{2}[J_1(ix) + iN_1(ix)] \tag{43}$$

where $H_1^{(1)}$ is a Hankel function, J_1 is a Bessel function of order 1, and N_1 is a Neumann function of order 1. Keep the first few terms in the series expansion(small argument),[6] and find

$$K_1(x) \approx \frac{1}{x} + \frac{x}{2}\ln\left(\frac{x}{2}\right) + \frac{x}{2}\left(\gamma_E - \frac{1}{2}\right) + ... \tag{44}$$

where $\gamma_E = .5772...$ is the Euler constant. Then

$$\Sigma_2'(\not p = m_0, \mathbf{p} = 0) = -\frac{\alpha\, m_0}{\pi}\int_0^1 (1+t)dt\big[\ln{(m_0 a)} + \ln{(t)}) + \gamma_E - \frac{1}{2}\big] \tag{45}$$

where α(the fine structure constant)$=e^2/4\pi$. Integration yields after putting back the constants $\hbar$ and c

$$\Sigma_2'(\not p = m_0, \mathbf{p} = 0) = -\frac{\alpha\, m_0}{2\pi}\big[3\ln\big(\frac{m_0 c a}{\hbar}\big) + 3\gamma_E - 4\big]. \tag{46}$$

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