A NEW THEORY OF MUON-PROTON SCATTERING

ABSTRACT

The muon charge is considered to be distributed or extended in space. The differential of the muon charge is set equal to a function of muon charge coordinates multiplied by a four-dimensional differential volume element. The four-dimensional integral of this function is required to equal the muon charge in all Lorentz frames. The structure of the proton can be written in terms of two proton form factors. The S-matrix for the extended muon and structured proton is calculated. The result is that the S-matrix contains three form factors. When the proton is treated as structured-extended, the resulting S-matrix acquires a fourth form factor. The calculations can also be applied to electron-proton scattering.

I. INTRODUCTION

In the rest frame of a muon charge distribution, let $x_r'^{\mu} = (x_r'^0, x_r'^1, x_r'^2, x_r'^3)$ denote a spacetime charge point, and let $x_r^{\mu} = (x_r^0, x_r^1, x_r^2, x_r^3)$ denote

Date: April 29, 2015.

the center of the charge distribution. Sometimes the superscript on the four-vector (not the components) will be omitted, and we will write $x'_r = (x'^0_r, x'^1_r, x'^2_r, x'^3_r)$ and $x_r = (x^0_r, x^1_r, x^2_r, x^3_r)$. Introduce $\tilde{x}_r = x'_r - x_r$ or equivalently $\tilde{x}^\mu_r = x'^\mu_r - x^\mu_r$. In a frame of reference in which the muon charge distribution moves with a speed β in the $+x^3$ direction, let $x'_m = (x'^0_m, x'^1_m, x'^2_m, x'^3_m)$ denote a spacetime charge point, and let $x_m = (x^0_m, x^1_m, x^2_m, x^3_m)$ denote the center of the charge distribution. Introduce $\tilde{x}_m = x'_m - x_m$. A Lorentz transformation yields $\tilde{x}^1_r = \tilde{x}^1_m$, $\tilde{x}^2_r = \tilde{x}^2_m$, $\tilde{x}^3_r = \gamma(\tilde{x}^3_m - \beta\tilde{x}^0_m)$, and $\tilde{x}^0_r = \gamma(\tilde{x}^0_m - \beta\tilde{x}^3_m)$ where $\gamma = 1/\sqrt{1-\beta^2}$. Denote this Lorentz transformation by $\tilde{x}_r = L(\tilde{x}_m)$.

In the rest frame, the muon charge e is equal to $\int \rho_r(\tilde{x}_r)\delta(\tilde{x}^0_r)d^4\tilde{x}_r$ where $\rho_r(\tilde{x}_r)$ is the charge density in the rest frame and δ denotes the delta function.[1] In the m frame, the muon charge e is equal to $\int \rho_r(L(\tilde{x}_m))\delta[\gamma(\tilde{x}^0_m - \beta\tilde{x}^3_m)]d^4\tilde{x}_m$.[2] So an element of charge de_m in the m frame is given by

$$de_m = \rho_r(L(\tilde{x}_m))\delta[\gamma(\tilde{x}^0_m - \beta\tilde{x}^3_m)]d^4\tilde{x}_m. \tag{1}$$

The next section is a review of point muon scattering by a point proton. The third section will study scattering of a point muon by a structured proton. In the fourth section, the S-matrix of an extended muon and a structured proton is calculated. The result is that the

S-matrix equals the S-matrix of point electron scattering by the structured proton multiplied by an electron form factor. Coordinates for an extended proton at rest are introduced in the next section. In the sixth section, the S-matrix for extended muon-extended and structured proton is calculated. The paper closes with a short comparison of this extended particle theory with conventional QED.

II. POINT MUON-POINT PROTON SCATTERING

The calculation for muon-proton scattering will follow the calculation of electron-proton scattering in Bjorken and Drell.[3] When the point muon at spacetime point x exchanges a photon with the point proton at spacetime point y, the S matrix element is approximated by

$$S_{fi} = \int d^4x\, d^4y\, \bar{\phi}_f(x)(-ie\gamma_\mu))\, \phi_i(x) i D_F(x-y)\phi_F(y)(-ie\gamma^\mu)\phi_I(y). \tag{2}$$

The initial exact muon wave function is approximated by the plane wave solution to the Dirac equation. The plane wave solution, which is normalized to unity in a box of volume V, is

$$\phi_i(x) = \sqrt{\frac{m}{E_i V}}\, u_i \exp\left(-ip_i \cdot x\right) \tag{3}$$

where $\hbar$ and c have been set equal to 1, m is the muon rest mass, u_i is a four-component spinor, which depends on the initial spin and on $p_i = (p_i^0 = E_i, p_i^1, p_i^2, p_i^3)$, the initial four-momentum, and γ^μ are the four Dirac matrices, which are labelled by $\mu = 0, 1, 2, 3$. The final muon wave function is

$$\phi_f(x) = \sqrt{\frac{m}{E_f V}}\, u_f \exp\left(-ip_f \cdot x\right), \tag{4}$$

where p_f is the final muon four-momentum, E_f is the final muon energy, u_f is the final muon spinor, and $\bar{\phi}_f = \phi_f^\dagger \gamma^0$. The photon propagator is

$$D_F(x-y) = \int \frac{d^4q}{(2\pi)^4} \exp\left[-iq \cdot (x-y)\right] \frac{-1}{q^2 + i\epsilon} \tag{5}$$

where q is the four-momentum of the photon, and the spacetime point y is the the argument of the proton wave function. The initial approximate proton wave function is

$$\phi_I(y) = \sqrt{\frac{M}{E_I V}}\, u_I \exp(-ip_I \cdot y) \tag{6}$$

where M is the proton mass, u_I is a four component spinor, which depends on the initial proton spin and on $p_I = (p_I^0 = E_I, p_I^1, p_I^2, p_I^3)$, the initial proton four-momentum. The final proton wave function is

$$\phi_F(y) = \sqrt{\frac{M}{E_F V}}\, u_F \exp(-ip_F \cdot y) \tag{7}$$

where p_F is the final proton momentum four-vector,E_F is the final proton energy, and u_F is the final four-component spinor of the proton. Substituting Eqs. (3), (4), (5), (6), and (7) into Eq. (2) yields

$$S_{fi} = \frac{+ie^2 mM(\bar{u}_f \gamma^\mu u_i)(\bar{u}_F \gamma_\mu u_I)}{(2\pi)^4 V^2 \sqrt{E_i E_f E_I E_F}} \int \frac{d^4x\, d^4y\, d^4q\, \exp[i(p_f - p_i - q)\cdot x]\exp[i(p_F - p_I + q)\cdot y]}{q^2 + i\epsilon}. \tag{8}$$

Perform the following integrations:

$$\int \exp(i(p_f - p_i - q)\cdot x) d^4x = (2\pi)^4 \delta^4(p_f - p_i - q); \tag{9}$$

$$\int \exp(i(p_F - p_I + q)\cdot y) d^4y = (2\pi)^4 \delta^4(p_F - p_I + q); \tag{10}$$

$$\int \delta^4(p_f - p_i - q)\delta^4(p_F - p_I + q)\frac{d^4q}{q^2 + i\epsilon} = \frac{\delta^4(p_F + p_f - p_I - p_i)}{(p_f - p_i)^2}; \tag{11}$$

and find

$$S_{fi} = \frac{+ie^2 mM}{V^2\sqrt{E_i E_f E_I E_F}}(2\pi)^4\delta^4(p_f + p_F - p_i - p_I)\frac{(\bar{u}_f\gamma^\mu u_i)(\bar{u}_F\gamma_\mu u_I)}{(p_f - p_i)^2}. \tag{12}$$

III. POINT MUON-STRUCTURED PROTON SCATTERING

The S-matrix for point electron-structured proton scattering is [4,5]

$$S_{fip} = \int d^4x\, d^4y\, \bar{\phi}_f(x)(-ie\gamma_\mu))\, \phi_i(x) i D_F(x-y) \\ \phi_F(y)(+ie)\Big(\gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2M}F_2(q^2)\Big)\phi_I(y). \tag{13}$$

where $q = p_f - p_i$, $\sigma^{\mu\nu} = i[\gamma^\mu, \gamma^\nu]/2$, and F_1 and F_2 are proton form factors, which are to be determined from experiment. The above equation also applies to muon-proton scattering. Substituting Eqs. (3), (4), (5), (6), and (7) into Eq. (13) yields

$$S_{fip} = \frac{+ie^2 mM}{V^2\sqrt{E_i E_f E_I E_F}}\frac{(2\pi)^4\delta^4(p_f + p_F - p_i - p_I)}{(p_f - p_i)^2} \\ (\bar{u}_f\gamma_\mu u_i)\Big(\bar{u}_F\big(\gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2M}F_2(q^2)\big)u_I\Big). \tag{14}$$

IV. EXTENDED MUON-STRUCTURED PROTON SCATTERING

In the previous section, x was the argument of the muon wave function and the muon charge spacetime point in an arbitrary Lorentz frame. Now take the muon to be initally moving with a speed β in the $+x^3$ direction. This previously was called the m frame. So now let x_m be the argument of the wave function and also the center of the muon charge distribution, and x'_m denotes spacetime charge point. Introduce $x'_m = x_m + \tilde{x}_m$.

Suppose that the proton is initially at rest in the m frame. Let $y_r = (y_r^0, y_r^1, y_r^2, y_r^3)$ denote the argument of the proton wave function in the m frame The subscript r is attached to y to emphasize that the proton is at rest in the m frame.

Eq. (13) will be modified to take into account the spatial charge distribution of the muon. The interaction takes place at charge points, so replace $D_F(x-y)$ by $D_F(x'_m - y_r)$. In addition, the muon charge is replaced by the four-dimensional integral of de_m. Then the S-matrix is

$$S_{fip\mu} = \int d^4x_m \, d^4y_r \, \bar{\phi}_f(x_m)(-ide_m\gamma_\mu)) \, \phi_i(x_m) iD_F(x'_m - y_r)$$

$$\phi_F(y_r)(+ie)\Big(\gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2M}F_2(q^2)\Big)\phi_I(y_r). \quad (15)$$

Use $D_F(x'_m - y_r) = D_F(x_m - y_r)\exp(-iq_m \cdot \tilde{x}_m)$, and $q_m = p_f - p_i$ as measured in the m frame. Then

$$S_{fip\mu} = S_{fip}F(q) \tag{16}$$

where the muon form factor $F(q)$ is given by

$$F(q) = \int \exp(-iq_m \cdot \tilde{x}_m)\frac{de_m}{e} \tag{17}$$

For mathematical convenience, pick

$$\rho_r(\tilde{x}_r) = \frac{e\delta(\sqrt{(\tilde{x}_r^1)^2 + (\tilde{x}_r^2)^2 + (\tilde{x}_r^3)^2} - a)}{4\pi a^2}, \tag{18}$$

so

$$\rho_r(L(\tilde{x}_m)) = \frac{e\delta(\sqrt{(\tilde{x}_m^1)^2 + (\tilde{x}_m^2)^2 + \gamma^2(\tilde{x}_m^3 - \beta\tilde{x}_m^0)^2} - a)}{4\pi a^2}. \tag{19}$$

Then

$$F(q) = \int \exp(-iq_m^0\tilde{x}_m^0 + iq_m^1\tilde{x}_m^1 + iq_m^1\tilde{x}_m^1 + iq_m^1\tilde{x}_m^1)$$
$$\frac{\delta(\sqrt{(\tilde{x}_m^1)^2 + (\tilde{x}_m^2)^2 + \gamma^2(\tilde{x}_m^3 - \beta\tilde{x}_m^0)^2} - a)}{4\pi a^2}\delta[\gamma(\tilde{x}_m^0 - \beta\tilde{x}_m^3)]d^4\tilde{x}_m. \tag{20}$$

Integration yields $F(q) = j_0(\hat{q}_m a)$. Since $q_m = p_f - p_i$, and $\beta = p_i^3/p_i^0$,

$$(\hat{q}_m)^2 = (p_f^1)^2 + (p_f^2) + \gamma^2(p_f^3 - \beta p_f^0)^2 = \frac{(p_f \cdot p_i)^2}{m^2} - m^2.^{\,2} \tag{21}$$

V. EXTENDED PROTON COORDINATES

In the rest frame of the proton, let $y_r'^{\mu} = (y_r'^0, y_r'^1, y_r'^2, y_r'^3)$ denote a spacetime charge point, and let $y_r^{\mu} = (y_r^0, y_r^1, y_r^2, y_r^3)$ denote the center of the charge distribution. Introduce $\tilde{y}_r = y_r' - y_r$ or equivalently $\tilde{y}_r^{\mu} = y_r'^{\mu} - y_r^{\mu}$.

In the rest frame, the proton charge $-e$ is equal to $\int \rho_r(\tilde{y}_r)\delta(\tilde{y}_r^0)d^4\tilde{y}_r$ where $\rho_r(\tilde{y}_r)$ is the charge density in the rest frame.[1] Thus an element of proton charge in the rest frame is

$$d(-e_r) = \rho_r(\tilde{y}_r)\delta(\tilde{y}_r^0)d^4\tilde{y}_r. \tag{22}$$

Pick

$$\rho_r(\tilde{y}_r) = \frac{e\delta(\sqrt{(\tilde{y}_r^1)^2 + (\tilde{y}_r^2)^2 + (\tilde{y}_r^3)^2} - a_p)}{4\pi a_p^2} \tag{23}$$

where a_p is the proton radius.

VI. EXTENDED MUON-EXTENDED STRUCTURED PROTON SCATTERING

Eq. (13) is now modified to take into account the spatial charge distributions of the muon and the proton. Since the interaction takes place at charge points, $D_F(x-y)$ is replaced by $D_F(x'_m - y'_r)$. In addition, the muon charge is replaced by the four-dimensional integral of de_m, and the proton charge is replaced by the four-dimensional integral of $d(-e_r)$. Then the S-matrix is

$$S_{fiep\mu} = \int d^4x_m\, d^4y_r\, \bar{\phi}_f(x_m)(-ide_m\gamma_\mu))\, \phi_i(x_m) iD_F(x'_m - y'_r)$$
$$\phi_F(y_r)(-id(-e_r)\Big(\gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2M}F_2(q^2)\Big)\phi_I(y_r). \quad (24)$$

Upon making the substitutions as done earlier

$$S_{fiep\mu} = \int d^4x_m\, d^4y_r\, \bar{\phi}_f(x_m)(-ie\gamma_\mu))\, \phi_i(x_m) iD_F(x_m - y_r)$$
$$\phi_F(y_r)(+ie)\Big(\gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2M}F_2(q^2)\Big)\phi_I(y_r)$$
$$\exp\left(-iq\cdot\tilde{x}_m\right)\frac{\rho_r(\tilde{x}_m)}{e}\delta[\gamma(\tilde{x}^0_m - \beta\tilde{x}^3_m)]d^4\tilde{x}_m$$
$$\exp\left(+iq\cdot\tilde{y}_r\right)\frac{\rho_{rp}(\tilde{y}_r)}{-e}\delta(\tilde{y}^0_r)d^4\tilde{y}_r. \quad (25)$$

This equation can be written

$$S_{fiep\mu} = S_{fip}F(q)F_p(q). \quad (26)$$

where as before the muon form factor $F(q) = j_0(\hat{q}_m a)$. The proton form factor $F_p(q)$ is given by

$$F_p(q) = \int \exp\left(+iq \cdot \tilde{y}_r\right) \frac{\rho_{rp}(\tilde{y}_r)}{-e} \delta(\tilde{y}_r^0) d^4\tilde{y}_r = j_0(|\mathbf{q}|a_p). \tag{27}$$

It might be possible to separate the effect of proton size from strong interaction effects.

VII. DISCUSSION OF THE THEORY

It has been shown that muon size alters the S-matrix for muon-proton scattering. The S-matrix for extended muon-proton scattering is equal to the S-matrix for point muon-proton scattering times $j_0(\hat{q}_m a)$ for the chosen muon charge density. The muon radius a is expected to be small, so $j_0(\hat{q}_m a) \approx 1 - (\hat{q}_m a)^2/3!$. Thus the S-matrix for the extended muon will be smaller than the S-matrix for the point muon as calculated in QED. Also, there will be an additional dependence on the scattering angle through $\hat{q}_m$. Thus the theory makes predictions that can be compared with experiment. The calculations in this paper can also be applied to electron-proton scattering.

ACKNOWLEDGEMENTS

I thank Ben for his many improvements to the paper.

References

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