**FINITE SELF MASS OF THE ELECTRON II**

ABSTRACT

The charge of the electron is postulated to be distributed or extended in space. The differential of the electron charge is set equal to a function of electron charge coordinates multiplied by a four-dimensional differential volume element. The four-dimensional integral of this function is required to equal the electron charge in all Lorentz frames. The self-mass of such an electron is found to be finite.

## I. LOGARITHMIC DIVERGENCE OF THE SELF-MASS

This section will show that the self-mass of a point electron diverges logarithmically. The electron self-mass is associated with the following sequence of events: a point electron propagates from spacetime point $y$ to spacetime point $w$ where it emits a photon, the electron then propagates to spacetime point $z$ where it absorbs the same photon, and finally the electron then propagates to spacetime point $x$. The propagator for this process, which is second order in the charge $e$,[1] is

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$$S_{F2}(x-y) = \int S_F(x-z)(-ie\gamma^\mu)S_F(z-w)(-ie\gamma_\mu)$$

$$S_F(w-y)D_F(z-w)d^4z\,d^4w \quad (1)$$

where $\gamma^\mu$ and $\gamma_\mu$ $(\mu = 0,1,2,3)$ both represent the Dirac matrices, $S_F(x-z), S_F(z-w)$, and $S_F(w-y)$ represent electron propagators. The electron propagators are given by

$$S_F(x-z) = \int \exp\left(-ip'\cdot(x-z)\right)S_F(p')\frac{d^4p'}{(2\pi)^4}\,, \quad (2)$$

$$S_F(z-w) = \int \exp\left(-ik\cdot(z-w)\right)S_F(k)\frac{d^4k}{(2\pi)^4}\,, \quad (3)$$

and

$$S_F(w-y) = \int \exp\left(-ip\cdot(w-y)\right)S_F(p)\frac{d^4p}{(2\pi)^4}\,, \quad (4)$$

where we choose the following representations in momentum space for the electron propagators:

$$S_F(p') = \frac{i(\not{p}' + m_0)}{p'^2 - m_0^2}, \quad (5)$$

$$S_F(p) = \frac{i(\not{p} + m_0)}{p^2 - m_0^2}, \quad (6)$$

and

$$S_F(k) = \frac{i(\not{k} + m_0)}{2\sqrt{\mathbf{k}^2 + m_0^2}}\Big(\frac{1}{k^0 - \sqrt{\mathbf{k}^2 + m_0^2} + i\epsilon} - \frac{1}{k^0 + \sqrt{\mathbf{k}^2 + m_0^2} - i\epsilon}\Big) \tag{7}$$

where $p, p'$, and $k$ are four-momentum vectors, $\not{p}' = \gamma^\mu p'_\mu$, $\not{p} = \gamma^\mu p_\mu$, $\not{k} = \gamma^\mu k_\mu$, and $m_0$ is called the bare electron mass. The observable or physical mass is given by $m = m_0 + \delta m$ where $\delta m$ is the self-mass. The photon propagator is given by

$$D_F(z - w) = -i \int \exp\left(-iq \cdot (z - w)\right) D_F(q) \frac{d^4q}{(2\pi)^4}. \tag{8}$$

Choose the following representation for the photon propagator in momentum space:

$$D_F(q) = \frac{1}{2|\mathbf{q}|}\Big(\frac{1}{q^0 - |\mathbf{q}| + i\epsilon} - \frac{1}{q^0 + |\mathbf{q}| - i\epsilon}\Big) \tag{9}$$

where $q$ is a four-momentum vector. It should be noted that the Eq. (7) and Eq. (9) are not the usual choice for the propagators.

Introduce the following abbreviations:

$$\Big(P\Big) = S_F(p) \exp\left(+ip \cdot y\right) \frac{d^4p}{(2\pi)^4}, \tag{10}$$

and

$$\Big(P'\Big) = S_F(p')\exp\left(-ip'\cdot x\right)\frac{d^4p'}{(2\pi)^4}. \tag{11}$$

Then using $\gamma^\mu(\not{k}+m)\gamma_\mu = -2\not{k} + 4m_0$.

$$S_{F2}(x-y) = (-ie)^2\int\Big(P'\Big)\exp\left(ip'\cdot z\right)\frac{i(-2\not{k}+4m_0)}{2\sqrt{\mathbf{k}^2+m_0^2}}\exp\left(-ik\cdot(z-w)\right)$$
$$\frac{d^4k}{(2\pi)^4}\Big(\frac{1}{k^0-\sqrt{\mathbf{k}^2+m_0^2}+i\epsilon}-\frac{1}{k^0+\sqrt{\mathbf{k}^2+m_0^2}-i\epsilon}\Big)$$
$$(-i)\frac{d^4q}{(2\pi)^4}\frac{\exp\left(-iq\cdot(z-w)\right)}{2|\mathbf{q}|}\Big(\frac{1}{q^0-|\mathbf{q}|+i\epsilon}-\frac{1}{q^0+|\mathbf{q}|-i\epsilon}\Big)$$
$$d^4z\,d^4w\,\exp\left(-ip\cdot w\right)\Big(P\Big)$$
$$=\int\Big(P'\Big)\exp\left(ip'^0z^0\right)I_1I_2\exp\left(-ip^0w^0\right)dz^0dw^0I_3\Big(P\Big). \tag{12}$$

where

$$I_1 = \int\frac{dk^0}{(2\pi)}\exp(-ik^0(z^0-w^0))\frac{(-2\not{k}+4m_0)}{2\sqrt{\mathbf{k}^2+m_0^2}}$$
$$\Big(\frac{1}{k^0-\sqrt{\mathbf{k}^2+m_0^2}+i\epsilon}+\frac{-1}{k^0+\sqrt{\mathbf{k}^2+m_0^2}-i\epsilon}\Big), \tag{13}$$

$$I_2 = \int\frac{dq^0}{(2\pi)}\frac{\exp\left(-iq^0(z^0-w^0)\right)}{2|\mathbf{q}|}\Big(\frac{1}{q^0-|\mathbf{q}|+i\epsilon}-\frac{1}{q^0+|\mathbf{q}|-i\epsilon}\Big), \tag{14}$$

and

$$I_3 = \int \exp\left[i(\mathbf{k}+\mathbf{q}-\mathbf{p}')\cdot \mathbf{z}\right]d^3z \exp\left[i(-\mathbf{k}-\mathbf{q}+\mathbf{p})\cdot \mathbf{w}\right]d^3w$$

$$= (2\pi)^3\delta^3(\mathbf{k}+\mathbf{q}-\mathbf{p}')(2\pi)^3\delta^3(-\mathbf{k}-\mathbf{q}+\mathbf{p})$$

$$= (2\pi)^3\delta^3(\mathbf{p}'-\mathbf{p})(2\pi)^3\delta^3(-\mathbf{k}-\mathbf{q}+\mathbf{p}). \quad (15)$$

Here $\delta()$ denotes the delta function. Use contour integration and find

$$I_1 = -iH(z^0-w^0)\exp(-i\sqrt{\mathbf{k}^2+m_0^2}\,(z^0-w^0))\frac{(-2\gamma^0\sqrt{\mathbf{k}^2+m_0^2}-2\gamma^j k_j+4m_0)}{2\sqrt{\mathbf{k}^2+m_0^2}}$$
$$-iH(w^0-z^0)\exp(+i\sqrt{\mathbf{k}^2+m_0^2}\,(z^0-w^0))\frac{(+2\gamma^0\sqrt{\mathbf{k}^2+m_0^2}-2\gamma^j k_j+4m_0)}{2\sqrt{\mathbf{k}^2+m_0^2}}, \quad (16)$$

where H() is the unit step function, and the index $j = 1, 2, 3$. Again use contour integration and find

$$I_2 = -i\frac{\exp\left(-i|\mathbf{q}|(z^0-w^0)\right)}{2|\mathbf{q}|}H(z^0-w^0)$$
$$-i\frac{\exp\left(-i|\mathbf{q}|(w^0-z^0)\right)}{2|\mathbf{q}|}H(w^0-z^0). \quad (17)$$

The product $I_1I_2$ contains four terms. Since $H(z^0-w^0)H(w^0-z^0) = 0$, two of the terms are zero. So

$$
\begin{aligned}
I_1 I_2 = (-i)^2 \Big( & \frac{-2\gamma^0 \sqrt{\mathbf{k}^2+m_0^2} - 2\gamma^j k_j + 4m_0}{2|\mathbf{q}|\, 2\sqrt{\mathbf{k}^2+m_0^2}} \Big) \\
& H(z^0 - w^0) \exp\big(-i(\sqrt{\mathbf{k}^2+m_0^2} + |\mathbf{q}|)(z^0 - w^0)\big) \\
& + (-i)^2 \Big( \frac{+2\gamma^0 \sqrt{\mathbf{k}^2+m_0^2} - 2\gamma^j k_j + 4m_0}{2|\mathbf{q}|\, 2\sqrt{\mathbf{k}^2+m_0^2}} \Big) \\
& H(w^0 - z^0) \exp\big(-i(\sqrt{\mathbf{k}^2+m_0^2} + |\mathbf{q}|)(w^0 - z^0)\big). \quad (18)
\end{aligned}
$$

Introduce

$$
\begin{aligned}
I_1' = \int & \exp\big((-i(\sqrt{\mathbf{k}^2+m_0^2} + |\mathbf{q}|)(z^0 - w^0)\big) H(z^0 - w^0) \\
& \exp\,[i({p'}^0 z^0 - p^0 w^0)] dz^0 dw^0\,, \quad (19)
\end{aligned}
$$

and

$$
\begin{aligned}
I_2' = \int & \exp\big((+i(\sqrt{\mathbf{k}^2+m_0^2} + |\mathbf{q}|)(z^0 - w^0)\big) H(w^0 - z^0) \\
& \exp\,[i({p'}^0 z^0 - p^0 w^0)] dz^0 dw^0\,. \quad (20)
\end{aligned}
$$

In Eq. (19), change variables from $z^0$ to $t^0 = z^0 - w^0$, so

$$
I_1' = \int H(t^0) \exp\big(-i(\sqrt{\mathbf{k}^2+m_0^2} + |\mathbf{q}| - p'^0)t^0\big) dt^0 \exp\,[i({p'}^0 - p^0) w^0] dw^0\,. \quad (21)
$$

By the theory of generalized functions,[2] $I_1' = 2\pi\, \delta({p'}^0 - p^0) I_{S1}$ where

$$I_{S1} = \Big(\frac{-i}{\sqrt{\mathbf{k}^2+m_0^2}+|\mathbf{q}|-p'^0} + \pi\delta(\sqrt{\mathbf{k}^2+m_0^2}+|\mathbf{q}|-p'^0)\Big). \qquad (22)$$

In Eq. (20), change variables from $w^0$ to $t^0 = w^0 - z^0$, so

$$I_2' = \int H(t^0)\exp\big(-i(\sqrt{\mathbf{k}^2+m_0^2}+|\mathbf{q}|+p^0)t^0\big)dt^0 \exp\left[i(p'^0-p^0)z^0\right]dz^0\,. \qquad (23)$$

By the theory of generalized functions,[2] $I_2' = 2\pi\,\delta(p'^0-p^0)I_{S2}$ where

$$I_{S2} = \Big(\frac{-i}{\sqrt{\mathbf{k}^2+m_0^2}+|\mathbf{q}|+p^0} + \pi\delta(\sqrt{\mathbf{k}^2+m_0^2}+|\mathbf{q}|+p^0)\Big). \qquad (24)$$

A delta function vanishes when its argument is not zero. The argument of the delta functions in Eqs. (22) and (24) is never zero, so both delta functions are zero and may be dropped. Then

$$\begin{aligned}
S_{F2}(x-y) &= -e^2\int \Big(P'\Big)\big(I_1'(-2\gamma^0\sqrt{\mathbf{k}^2+m_0^2}-2\gamma^j k_j+4m_0)\big) \\
&\quad + \big(I_2'(+2\gamma^0\sqrt{\mathbf{k}^2+m_0^2}-2\gamma^j k_j+4m_0)\big)I_3\frac{d^3k}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\Big)\Big(P\Big) \\
&= -e^2\int\Big(P'\Big)\frac{d^3k}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\frac{1}{2|\mathbf{q}|2\sqrt{\mathbf{k}^2+m_0^2}} \\
&\quad (2\pi)^3\delta^3(\mathbf{p}'-\mathbf{p})(2\pi)^3\delta^3(-\mathbf{k}-\mathbf{q}+\mathbf{p}) \\
&\quad 2\pi\,\delta(p'^0-p^0)\Big[I_{S1}(-2\gamma^0\sqrt{\mathbf{k}^2+m_0^2}-2\gamma^j k_j+4m_0)+ \\
&\quad I_{S2}(+2\gamma^0\sqrt{\mathbf{k}^2+m_0^2}-2\gamma^j k_j+4m_0)\Big]\Big(P\Big). \qquad (25)
\end{aligned}$$

Use

$$\int \frac{d^4p'}{(2\pi)^4} S_F(p') \exp(-ip'\cdot x)(2\pi)^4\delta^4(p'-p) = S_F(p)\exp(-ip\cdot x), \tag{26}$$

and find

$$\begin{aligned} S_{F2}(x-y) = -ie^2 \int S_F(p)\exp(-ip\cdot x)\frac{(2\pi)^3\delta^3(+\mathbf{k}+\mathbf{q}-\mathbf{p})}{2|\mathbf{q}|\,2\sqrt{\mathbf{k}^2+m_0^2}}\frac{d^3k}{(2\pi)^3}\frac{d^3q}{(2\pi)^3} \\ \Big[I_{S1}(-2\gamma^0\sqrt{\mathbf{k}^2+m_0^2} - 2\gamma^j k_j + 4m_0) + \\ I_{S2}(+2\gamma^0\sqrt{\mathbf{k}^2+m_0^2} - 2\gamma^j k_j + 4m_0)\Big] S_F(p)\exp(ip\cdot y)\frac{d^4p}{(2\pi)^3} \\ = \int S_F(p)\exp(-ip\cdot x)(-i\Sigma_2(p))S_F(p)\exp(+ip\cdot y)\frac{d^4p}{(2\pi)^4} \end{aligned} \tag{27}$$

where

$$\begin{aligned} \Sigma_2(p) = e^2 \int \frac{d^3k}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}(2\pi)^3\delta^3(\mathbf{k}+\mathbf{q}-\mathbf{p})\frac{1}{2|\mathbf{q}|\,2\sqrt{\mathbf{k}^2+m_0^2}} \\ \Big[I_{S1}(-2\gamma^0\sqrt{\mathbf{k}^2+m_0^2} - 2\gamma^j k_j + 4m_0) + \\ I_{S2}(+2\gamma^0\sqrt{\mathbf{k}^2+m_0^2} - 2\gamma^j k_j + 4m_0)\Big]. \end{aligned} \tag{28}$$

$\Sigma_2(\not p = m_0)$ is identified approximately as $\delta m$ the self-mass.[1]

It is convenient to continue the calculation in the frame where the electron is initially at rest, so $\mathbf{p} = 0$ and $\mathbf{k} = -\mathbf{q}$. Then

$$\Sigma_2(\not p = m_0, \mathbf{p} = 0) = e^2 \int \frac{d^3q}{(2\pi)^3} \frac{1}{2|\mathbf{q}|\, 2\sqrt{\mathbf{q}^2 + m_0^2}}$$
$$\Big[\frac{-2\gamma^0\sqrt{\mathbf{q}^2 + m_0^2} + 2\gamma^j q_j + 4m_0}{\sqrt{\mathbf{q}^2 + m_0^2} + |\mathbf{q}| - m_0} + \frac{2\gamma^0\sqrt{\mathbf{q}^2 + m_0^2} + 2\gamma^j q_j + 4m_0}{\sqrt{\mathbf{q}^2 + m_0^2} + |\mathbf{q}| + m_0}\Big]. \tag{29}$$

The terms in the numerator, which are linear in $q_1, q_2$, and $q_3$, integrate to zero. Put the terms in brackets over a common denominator. Then

$$\Sigma_2(\not p = m_0, \mathbf{p} = 0) = \frac{e^2}{(2\pi)^3} \int d^3q \Big[\frac{-4m_0\sqrt{\mathbf{q}^2 + m_0^2} + 8m_0(|\mathbf{q}| + \sqrt{\mathbf{q}^2 + m_0^2}\,)}{2|\mathbf{q}|\, 2\sqrt{\mathbf{q}^2 + m_0^2}\, 2|\mathbf{q}|(|\mathbf{q}| + \sqrt{\mathbf{q}^2 + m_0^2}\,)}\Big]$$
$$= \frac{e^2 m_0}{4\pi^2} \int d|\mathbf{q}| \Big[-\frac{1}{|\mathbf{q}| + \sqrt{\mathbf{q}^2 + m_0^2}} + \frac{2}{\sqrt{\mathbf{q}^2 + m_0^2}}\Big] \tag{30}$$

Thus $\Sigma_2(\not p = m_0, \mathbf{p} = 0) = \delta m \to \int^\infty d|\mathbf{q}|/|\mathbf{q}| \to \log|\mathbf{q}|$ as $|\mathbf{q}| \to \infty$, and the self-mass of the point electron is logarithmically divergent.

## II. EXTENDED ELECTRON COORDINATES

In the rest frame of an electron charge distribution, let $x_r'^\mu = (x_r'^0, x_r'^1, x_r'^2, x_r'^3)$ denote a spacetime charge point. Let $x_r^\mu = (x_r^0, x_r^1, x_r^2, x_r^3)$ denote the center of the charge distribution. The charge distribution of the electron is assumed to have a well-defined center, which is identified as the argument of the wave function. The shape of the charge distribution depends on the motion of the charge, and is assumed to be unaffected by any interaction. Sometimes the superscript will be omitted, and we will write $x'_r = (x'^0_r, x'^1_r, x'^2_r, x'^3_r)$ and $x_r = (x^0_r, x^1_r, x^2_r, x^3_r)$. Introduce $\tilde{x}_r = x'_r - x_r$ or equivalently $\tilde{x}^\mu_r = x'^\mu_r - x^\mu_r$. In a frame of reference in which the charge distribution moves with a constant speed $\beta$ in the $+x^3$ direction, let $x'_m = (x'^0_m, x'^1_m, x'^2_m, x'^3_m)$ denote a spacetime charge point, and let $x_m = (x^0_m, x^1_m, x^2_m, x^3_m)$ denote the center of the charge distribution. Introduce $\tilde{x}_m = x'_m - x_m$. A Lorentz transformation yields $\tilde{x}^1_r = \tilde{x}^1_m$, $\tilde{x}^2_r = \tilde{x}^2_m$, $\tilde{x}^3_r = \gamma(\tilde{x}^3_m - \beta\tilde{x}^0_m)$, and $\tilde{x}^0_r = \gamma(\tilde{x}^0_m - \beta\tilde{x}^3_m)$ where $\gamma = 1/\sqrt{1-\beta^2}$.

In the rest frame, the electron charge $e$ is equal to $\int \rho_r(\tilde{x}_r)\delta(\tilde{x}^0_r)d^4\tilde{x}_r$ where $\rho_r(\tilde{x}_r)$ is the charge density in the rest frame.[3] So an element of charge in the rest frame is given by

$$de(\tilde{x}_r) = \rho_r(\tilde{x}_r)\delta(\tilde{x}^0_r)\, d^4\tilde{x}_r. \tag{31}$$

In the moving frame described above, the element of charge is

$$de(\tilde{x}_m) = \rho_r(\tilde{x}_m)\delta(\gamma(\tilde{x}^0_m - \beta\tilde{x}^3_m)\, d^4\tilde{x}_m.^{4} \tag{32}$$

Consider a uniformly charged spherical surface of radius $a$, which is at rest, and is centered at $\mathbf{x} = (x^1_r, x^2_r, x^3_r)$. The charge density is given by

$$\rho_r(\tilde{x}_r) = \frac{e}{4\pi a^2}\delta(\sqrt{(\tilde{x}_r^1)^2 + (\tilde{x}_r^2)^2 + (\tilde{x}_r^3)^2} - a) \tag{33}$$

When the charge moves in the $+x^3$ direction with a speed $\beta$, the charge density is given by

$$\rho_r(L(\tilde{x}_m)) = \frac{e}{4\pi a^2}\delta(\sqrt{(\tilde{x}_m^1)^2 + (\tilde{x}_m^2)^2 + \gamma^2(\tilde{x}_m^3 - \beta\tilde{x}_m^0)^2} - a) \tag{34}$$

## III. FINITE SELF-MASS OF THE EXTENDED ELECTRON

Take the extended electron to be initially at rest. It will be assumed that when an element of electron charge emits a photon of four-momentum $q$, then the electron recoils with a speed $\beta = |\mathbf{k}|/k^0$. When an element of electron charge absorbs a photon of four-momentum $q$, then the electron acquires a four-momentum $p' = p$.

The photons are emitted and absorbed at charge points, so replace $D_F(z - w)$ by $D_F(z' - w')$. Each charge point propagates between emission and absorption of the photon with momentum $k$, so $S_F(z-w)$ is replaced by $S_F(z' - w')$. In addition, the first $e$ in Eq. (1) is replaced by the four-dimensional integral of $de(\tilde{z})$, and the second $e$ is replaced by the four-dimensional integral of $de(\tilde{w})$. Then Eq. (1) is replaced by

$$S_{F2e}(x-y) = \int S_F(x-z)(-ide(\tilde{z})\gamma^\mu)S_F(z'-w')(-ide(\tilde{w})\gamma_\mu)$$
$$S_F(w-y)D_F(z'-w')d^4z\, d^4w, \tag{35}$$

and Eq. (12) is replaced by

$$
\begin{aligned}
S_{F2e}(x-y) = (-ie)^2 \int \Big(P'\Big) \exp(ip'\cdot z)\frac{i(-2\not{k}+4m_0)}{2\sqrt{\mathbf{k}^2+m_0^2}} \exp(-ik\cdot(z'-w')) \\
\frac{d^4k}{(2\pi)^4}\Big(\frac{1}{k^0-\sqrt{\mathbf{k}^2+m_0^2}+i\epsilon} - \frac{1}{k^0+\sqrt{\mathbf{k}^2+m_0^2}-i\epsilon}\Big) \\
(-i)\frac{d^4q}{(2\pi)^4}\frac{\exp(-iq\cdot(z'-w'))}{2|\mathbf{q}|}\Big(\frac{1}{q^0-|\mathbf{q}|+i\epsilon} - \frac{1}{q^0+|\mathbf{q}|-i\epsilon}\Big) \\
\frac{de(\tilde{z})}{e}\frac{de(\tilde{w})}{e} d^4z\, d^4w\, \exp(-ip\cdot w)\Big(P\Big) \quad (36)
\end{aligned}
$$

Change variables from $z$ to $z'$ and from $w$ to $w'$. and get

$$
\begin{aligned}
S_{F2e}(x-y) = (-ie)^2 \int \Big(P'\Big) \exp(ip'\cdot z')\frac{i(-2\not{k}+4m_0)}{2\sqrt{\mathbf{k}^2+m_0^2}} \exp(-ik\cdot(z'-w')) \\
\frac{d^4k}{(2\pi)^4}\Big(\frac{1}{k^0-\sqrt{\mathbf{k}^2+m_0^2}+i\epsilon} - \frac{1}{k^0+\sqrt{\mathbf{k}^2+m_0^2}-i\epsilon}\Big) \\
(-i)\frac{d^4q}{(2\pi)^4}\frac{\exp(-iq\cdot(z'-w'))}{2|\mathbf{q}|}\Big(\frac{1}{q^0-|\mathbf{q}|+i\epsilon} - \frac{1}{q^0+|\mathbf{q}|-i\epsilon}\Big) \\
\exp(-ip'\cdot\tilde{z})\frac{de(\tilde{z})}{e}\exp(+ip\cdot\tilde{w})\frac{de(\tilde{w})}{e} d^4z'\, d^4w'\, \exp(-ip\cdot w')\Big(P\Big) \\
(37)
\end{aligned}
$$

In place of Eq. (18) for $I_1 I_2$ is

$$(I_1 I_2)_e = (-i)^2 \Big( \frac{-2\gamma^0 \sqrt{\mathbf{k}^2 + m_0^2} - 2\gamma^j k_j + 4m_0}{2|\mathbf{q}|\, 2\sqrt{\mathbf{k}^2 + m_0^2}} \Big)$$
$$H(z'^0 - w'^0) \exp\big(-i(\sqrt{\mathbf{k}^2 + m_0^2} + |\mathbf{q}|)(z'^0 - w'^0)\big)$$
$$+ (-i)^2 \Big( \frac{+2\gamma^0 \sqrt{\mathbf{k}^2 + m_0^2} - 2\gamma^j k_j + 4m_0}{2|\mathbf{q}|\, 2\sqrt{\mathbf{k}^2 + m_0^2}} \Big)$$
$$H(w'^0 - z'^0) \exp\big(-i(\sqrt{\mathbf{k}^2 + m_0^2} + |\mathbf{q}|)(w'^0 - z'^0)\big), \quad (38)$$

which is to be multiplied by $\exp(-ip' \cdot \tilde{z}) de(\tilde{z}) \exp(+ip \cdot \tilde{w}) de(\tilde{w})$.

For the purpose of illustration, pick the electron charge to be uniformly distributed on a spherical surface of radius $a$ in the electron rest frame. For $z'^0 - w'^0 > 0$, the electron is initially at rest, so

$$\exp(+ip \cdot \tilde{w}_r) \frac{de(\tilde{w}_r)}{e} = \int \exp(ip^0 \tilde{w}_r^0 - i\mathbf{p} \cdot \tilde{\mathbf{w}}_r) \frac{1}{4\pi a^2}$$
$$\delta(\sqrt{(\tilde{w}_r^1)^2 + (\tilde{w}_r^2)^2 + (\tilde{w}_r^3)^2} - a)\, \delta(\tilde{w}_r^0)\, d^4\tilde{w}_r = 1. \quad (39)$$

After emission of the photon, the electron travels with a speed $\beta = |\mathbf{k}|/k^0$. Choose $k$ to be in the direction of $\tilde{z}^3$. Then

$$exp(-ip' \cdot \tilde{z}_m)\frac{de(\tilde{z}_m)}{e} = \int \exp\left(-ip'^0\tilde{z}^0_m + i\mathbf{p}' \cdot \tilde{\mathbf{z}}_m\right)\frac{1}{4\pi a^2}$$
$$\delta(\sqrt{(\tilde{z}^1_m)^2 + (\tilde{z}^2_m)^2 + \gamma^2(\tilde{z}^3_m - \beta\tilde{z}^0_m)^2} - a)\delta(\gamma(\tilde{z}^0_m - \beta\tilde{z}^3_m))d^4\tilde{z}_m =$$
$$\int \exp\left(-ip'^0\beta\tilde{z}^3_m\right)\frac{1}{4\pi a^2}\delta(\sqrt{(\tilde{z}^1_m)^2 + (\tilde{z}^2_m)^2 + (\tilde{z}^3_m)^2/\gamma^2} - a)\frac{d^3\tilde{z}_m}{\gamma} =$$
$$\int \exp\left(-im_0\beta\gamma\hat{z}^3_m\right)\frac{1}{4\pi a^2}\delta(|\hat{\mathbf{z}_m}| - a)d^3\hat{z}_m =$$
$$\frac{\exp\left(i|\mathbf{q}|a\right) - \exp\left(-i|\mathbf{q}|a\right)}{2i|\mathbf{q}|a} = \frac{\sin\left(|\mathbf{q}|a\right))}{|\mathbf{q}|a} \quad (40)$$

where $\beta\gamma = |\mathbf{k}|/\sqrt{k^{0\,2} - |\mathbf{k}|^2} = |\mathbf{q}|/m_0$, $\tilde{z}^1_m = \hat{z}^1_m$, $\tilde{z}^2_m = \hat{z}^2_m$, $\tilde{z}^3_m/\gamma = \hat{z}^3_m$, and $p'^0 = m_0$.

For $w'^0 - z'^0 > 0$, the electron is finally at rest. After emission of the photon, the electron travels backward in time with a speed $\beta = |\mathbf{k}|/k^0$. So

$$\exp\left(-ip' \cdot \tilde{z}_r\right)\frac{de(\tilde{z}_r)}{e} = \int \exp\left(-ip'^0\tilde{z}^0_r + i\mathbf{p}' \cdot \tilde{\mathbf{z}}_r\right)\frac{1}{4\pi a^2}$$
$$\delta(\sqrt{(\tilde{z}^1_r)^2 + (\tilde{z}^2_r)^2 + (\tilde{z}^3_r)^2} - a)\delta(\tilde{z}^0_r)\, d^4\tilde{z}_r = 1, \quad (41)$$

and

$$exp(+ip\cdot\tilde{w}_m)\frac{de(\tilde{w}_m)}{e} = \int \exp{(+ip^0\tilde{w}^0_m - i\mathbf{p}\cdot\tilde{\mathbf{w}}_m)}\frac{1}{4\pi a^2}$$

$$\delta(\sqrt{(\tilde{w}^1_m)^2+(\tilde{w}^2_m)^2+\gamma^2(\tilde{w}^3_m-\beta\tilde{w}^0_m)^2}-a)\delta(\gamma(\tilde{w}^0_m-\beta\tilde{w}^3_m))d^4\tilde{w}_m =$$

$$\int \exp{(+ip^0\beta\tilde{w}^3_m)}\frac{1}{4\pi a^2}\delta(\sqrt{(\tilde{w}^1_m)^2+(\tilde{w}^2_m)^2+(\tilde{w}^3_m)^2/\gamma^2}-a)\frac{d^3\tilde{w}_m}{\gamma} =$$

$$\int \exp{(-im_0\beta\gamma\hat{w}^3_m)}\frac{1}{4\pi a^2}\delta(|\hat{\mathbf{w}}_m|-a)d^3\hat{w}_m =$$

$$\frac{\exp{(i|\mathbf{q}|a)}-\exp{(-i|\mathbf{q}|a)}}{2i|\mathbf{q}|a} = \frac{\sin{(|\mathbf{q}|a))}}{|\mathbf{q}|a} \qquad (42)$$

. So Eq. (30) is replaced by

$$\Sigma_{2e}(\not{p}=m_0,\mathbf{p}=0) = \frac{e^2}{(2\pi)^3}\int d^3q\Big[\frac{-4m_0\sqrt{\mathbf{q}^2+m_0^2}+8m_0(|\mathbf{q}|+\sqrt{\mathbf{q}^2+m_0^2}\,)}{2|\mathbf{q}|\,2\sqrt{\mathbf{q}^2+m_0^2}\,2|\mathbf{q}|(|\mathbf{q}|+\sqrt{\mathbf{q}^2+m_0^2}\,)}\Big]$$

$$\frac{\sin{(|\mathbf{q}|a))}}{|\mathbf{q}|a} =$$

$$\frac{e^2m_0}{4\pi^2}\int d|\mathbf{q}|\Big[-\frac{1}{|\mathbf{q}|+\sqrt{\mathbf{q}^2+m_0^2}}+\frac{2}{\sqrt{\mathbf{q}^2+m_0^2}}\Big]\frac{\sin{(|\mathbf{q}|a))}}{|\mathbf{q}|a}. \qquad (43)$$

The integrand is finite for small $|\mathbf{q}|$ since $\sin{(|\mathbf{q}|a)}/(|\mathbf{q}|a) = 1$ at $|\mathbf{q}| = 0$. Take the absolute value of the integrand and note that for large $|\mathbf{q}|$

$$\int^\infty \frac{d|\mathbf{q}|}{(|\mathbf{q}|)}\Big[+\frac{1}{|\mathbf{q}|+\sqrt{\mathbf{q}^2+m_0^2})}+\frac{2}{\sqrt{\mathbf{q}^2+m_0^2}}\Big] \to \int^\infty \frac{d|\mathbf{q}|}{|\mathbf{q}|^2}. \qquad (44)$$

Thus the integral converges absolutely, and the self-mass is finite for the assumed charge distribution. This suggests, but does not prove that the self-mass will be finite for a charge distributed over a finite region of space. The integral for the self-mass will be evaluated in the appendix.

## APPENDIX

Write Eq. (43) in the form

$$\Sigma_{2e}(\not p = m_0, \mathbf{p} = 0) = \frac{e^2 m_0}{4\pi^2} \int_0^\infty d|\mathbf{q}| \Big[ -\frac{1}{|\mathbf{q}| + \sqrt{\mathbf{q}^2 + m_0^2}} + \frac{2}{\sqrt{\mathbf{q}^2 + m_0^2}} \Big] \frac{\exp(i|\mathbf{q}|a) - \exp(-i|\mathbf{q}|a)}{2i|\mathbf{q}|a}. \quad (45)$$

Multiply the first term in the brackets by $(-|\mathbf{q}|+\sqrt{\mathbf{q}^2 + m_0^2})/(-|\mathbf{q}|+\sqrt{\mathbf{q}^2 + m_0^2})$. Then multiply $-\sqrt{\mathbf{q}^2 + m_0^2}/m_0^2$ by $\sqrt{\mathbf{q}^2 + m_0^2}/\sqrt{\mathbf{q}^2 + m_0^2}$. The result is

$$\Sigma_{2e}(\not p = m_0, \mathbf{p} = 0) = \frac{e^2 m_0}{4\pi^2 2ia} \int_0^\infty d|\mathbf{q}| \Big[ -\frac{|\mathbf{q}|}{m_0^2\sqrt{\mathbf{q}^2 + m_0^2}} + \frac{1}{|\mathbf{q}|\sqrt{\mathbf{q}^2 + m_0^2}} + \frac{1}{m_0^2} \Big] \big( \exp(i|\mathbf{q}|a) - \exp(-i|\mathbf{q}|a) \big). \quad (46)$$

Change variables in the terms multiplied by $\exp(i|\mathbf{q}|a)$ by replacing $|\mathbf{q}|$ by $-|\mathbf{q}|$, and find

$$\Sigma_{2e}(\not p = m_0, \mathbf{p} = 0) = \frac{e^2 m_0}{4\pi^2 2ia} (I_{12} + I_{13} + I_4) \quad (47)$$

where

$$I_{12} = \int_{-\infty}^{+\infty} d|\mathbf{q}| \exp(-i|\mathbf{q}|a) \Big( \frac{|\mathbf{q}|}{m_0^2 \sqrt{\mathbf{q}^2 + m_0^2}} \Big), \quad (48)$$

$$I_{13} = -\int_{-\infty}^{+\infty} d|\mathbf{q}| \exp(-i|\mathbf{q}|a)\Big(\frac{1}{|\mathbf{q}|\sqrt{\mathbf{q}^2 + m_0^2}}\Big), \tag{49}$$

and

$$I_4 = -\int_{-\infty}^{+\infty} d|\mathbf{q}| \exp(-i|\mathbf{q}|a)\,\frac{\mathrm{sgn}(q)}{m_0^2} = \frac{+2i}{m_0^2 a}\,.^{5} \tag{50}$$

Set $|\mathbf{q}| = m_0|\mathbf{q}'|$, and find

$$I_{12} = \int d|\mathbf{q}'| \exp(-i|\mathbf{q}'|m_0 a)\Big(\frac{|\mathbf{q}'|}{m_0\sqrt{\mathbf{q}'^2 + 1}}\Big), \tag{51}$$

and

$$I_{13} = -\int d|\mathbf{q}'| \exp(-i|\mathbf{q}'|a)\Big(\frac{1}{m_0|\mathbf{q}'|\sqrt{\mathbf{q}'^2 + 1}}\Big), \tag{52}$$

To integrate Eqns. (51) and (52), the following integrals will be used with the convolution:

$$I_1 = \int d|\mathbf{q}'| \exp(-i|\mathbf{q}'|m_0 a)\Big(\frac{1}{\sqrt{\mathbf{q}'^2 + 1}}\Big) = 2K_0(|m_0 a|),\ ^{6} \tag{53}$$

$$I_2 = \int d|\mathbf{q}'| \exp(-i|\mathbf{q}'|m_0 a)|\mathbf{q}'| = 2\pi i\delta'(m_0 a),\ ^{5} \tag{54}$$

and

$$I_3 = \int d|\mathbf{q}'| \exp(-i|\mathbf{q}'|m_0 a)\frac{1}{|\mathbf{q}'|} = -\pi i\,\mathrm{sgn}(m_0 a).\ ^{6} \tag{55}$$

Here $K_0$ is the modified Bessel function of order zero, which is given by

$$K_0(x) = \frac{\pi i}{2}[J_0(ix) + iN_0(ix)]\,.^{7} \tag{56}$$

Here $J_0$ is the Bessel function of order zero, and $N_0$ is the Neuman function of order zero. Use the convolution [6] to write

$$I_{12} = \frac{1}{2\pi m_0}\int 2K_0(\alpha)2\pi i\delta'(ma-\alpha)d\alpha = \frac{2i}{m_0}\int K_0'(\alpha)\delta(\alpha - ma)\, d\alpha\,, \tag{57}$$

and

$$I_{13} = \frac{-1}{2\pi m_0}\int 2K_0(\alpha)(-\pi i)\mathrm{sgn}(ma-\alpha)d\alpha = \frac{+i}{m_0}\int_0^{ma} 2K_0(\alpha)\, d\alpha\,. \tag{58}$$

$K_0$ can be written as a series. For small argument, $K_0$ takes the form

$$K_0(x) \approx -\ln\big(\frac{x}{2}\big) - \gamma - \frac{x^2}{4}\Big(\ln\big(\frac{x}{2}\big) + \gamma - 1\Big)\,,^{8} \tag{59}$$

where $\gamma = 0.57721...$ is Euler's constant. Then

$$K_0'(x) \approx -\frac{1}{x} - \frac{x}{2}\Big(\ln\big(\frac{x}{2}\big) + \gamma - \frac{1}{2}\Big)\,, \tag{60}$$

so

$$I_{12} = \frac{2i}{m_0}K_0'(ma) \approx \frac{2i}{m_0}\Big[-\frac{1}{m_0 a} - \frac{m_0 a}{2}\Big(\ln\big(\frac{x}{2}\big) + \gamma - \frac{1}{2}\Big)\Big]\,, \tag{61}$$

and

$$I_{13} \approx -\frac{2i}{m_0}\Big(m_0 a \ln\big(\frac{m_0 a}{2}\big) + m_0 a\big(\gamma - \frac{1}{2}\big)\Big) \tag{62}$$

So

$$I_{12} + I_{13} + I_4 \approx \frac{2i}{m_0}\Big[-\frac{3m_0 a}{2}\Big(\ln\big(\frac{m_0 a}{2}\big) + \gamma - \frac{5}{4}\Big)\Big]. \tag{63}$$

and finally

$$\Sigma_{2e}(\not p = m_0, \mathbf{p} = 0) \approx \frac{e^2 m_0}{4\pi 2\pi}\big[-3\ln\Big(\frac{m_0 a}{2}\Big) - 3\gamma + \frac{5}{2}\big]. \qquad (64)$$

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