EFFECT OF ELECTRON AND PROTON SIZE ON H-ATOM SPECTRA

ABSTRACT

In the usual calculation of the H-atom energy levels, the potential energy of a point proton at rest and a point electron at rest is used in the Schrodinger equation. In this paper, the potential energy is modified by treating the electron and proton charges as extended in space. The effect of electron and proton size on the 2S energy level is calculated.

I. ENERGY SHIFT OF 2S ENERGY LEVEL FOR EXTENDED PROTON

Take the proton charge to be uniformly distributed on a spherical shell of radius r_0 in the proton rest frame. The potential $\Phi(r)$ due to the proton is given by

$$\Phi(r) = \frac{e}{r}H(r - r_0) + \frac{e}{r_0}H(r_0 - r) \tag{1}$$

where $+e$ is the proton charge, r is the distance of a field point from the center of the proton, and $H()$ is the unit step function which is

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given by H(x)=1 for $x > 1$ and $H(x) = 0$ for $x < 0$. The potential energy $V(r)$ of a point electron at $\mathbf{r}$ and the extended proton system is the electron charge $-e$ times the potential $\Phi(r)$, so

$$V(r) = \frac{-e^2}{r}H(r - r_0) + \frac{-e^2}{r_0}H(r_0 - r). \tag{2}$$

Add and subtract $(-e^2/r)H(r_0 - r)$, and get

$$V(r) = -\frac{e^2}{r} - \frac{e^2}{r_0}H(r_0 - r) + \frac{e^2}{r}H(r_0 - r) \tag{3}$$

Subtract $-e^2/r$ from $V(r)$ to get the perturbing potential energy δV. Thus

$$\delta V(r) = \frac{-e^2}{r_0}H(r_0 - r) + \frac{e^2}{r}H(r_0 - r). \tag{4}$$

The energy shift δE due to proton size is given by perturbation theory as

$$\delta E = \int \psi^*_{nlm}(r, \theta, \phi)\, \delta V(r)\, \psi_{nlm}(r, \theta, \phi)\, d^3r \tag{5}$$

where r, θ, and ϕ are the usual spherical coordinates, and $\psi_{nlm}(r, \theta, \phi)$ is the solution to the Schroedinger equation, which is written as $\psi_{nlm}(r, \theta, \phi) = R_{nl}(r)Y_{lm}(\theta, \phi)$. See. for example. reference 1 for the solutions to the hydrogen atom. For the 2S energy level, the normalized spherical harmonic $Y_{00} = 1/(4\pi)$, and

$$R_{20}(r) = \frac{1}{(2a_0)^{3/2}}\left(2 - \frac{r}{a_0}\right) \exp\left(-r/(2a_0)\right) \tag{6}$$

where a_0 is the Bohr radius. Expand the exponential Eq. (6) in a power series in $r/(2a_0) << 1$. Approximate R_{20}^2 by $(4-8r/a_0)/(2a_0)^3$, and carry out the integration in Eq. (5) for the 2S energy level. The result is

$$\delta E_{20} = \frac{2}{3}\frac{e^2}{(2a_0)^3} r_0^2 (1 - \frac{r_0}{a_0}). \tag{7}$$

See, for example, references 2 and 3 for how nuclear size contributes to the Lamb shift.

II. ENERGY SHIFT OF 2S ENERGY LEVEL FOR AN EXTENDED ELECTRON

Take electron charge to be uniformly distributed on a spherical shell of radius a in the electron rest frame. Let r' be the distance from the center of the proton charge to an element of the electron charge. For $r' > r_0$, the potential due to the proton is e/r'. When $r' < r_0$, the potential at r' is e/r_0. Let r be the distance from the center of the proton charge to the center of the electron charge. For all of the electron outside of the proton ($r > r_0 + a$), the potential energy is $-e^2/r$. For all of the electron inside of the proton ($r < r_0 - a$), the potential energy is $-e^2/r_0$. For $r_0 - a < r < r_0 + a$, part of the electron is inside the proton shell, and part is outside. Let $V_b(r)$ be the potential energy for $r_0 - a < r < r_0 + a$. Then

$$V(r) = \frac{-e^2}{r_0} H(r_0 - a - r) - \frac{e^2}{r} H(r - r_0 - a) + V_b[H(r - r_0 + a) - H(r - r_0 - a)]. \tag{8}$$

Add and subtract $(-e^2/r)H(r_0 + a - r))$, and get

$$V(r) = -\frac{e^2}{r} - \frac{e^2}{r_0} H(r_0 - a - r) + \frac{e^2}{r} H(r_0 + a - r) + V_b[H(r - r_0 + a) - H(r - r_0 - a)]. \tag{9}$$

Subtract $-e^2/r$ from V to get the perturbing potential energy $\delta V(r)$. Then

$$\delta V(r) = \frac{-e^2}{r_0} H(r_0 - a - r) + \frac{e^2}{r} H(r_0 + a - r) + V_b[H(r - r_0 + a) - H(r - r_0 - a)]. \tag{10}$$

Proceed as in the previous section, and find for the approximate energy shift of the 2S energy level

$$\delta E_{20} = \int \psi^*_{200}(r, \theta, \phi) \delta V(r)\, \psi_{200}(r, \theta, \phi) d^3 r = \int R^2_{20}(r) \delta V(r)\, r^2\, dr = \delta E^1_{20} + \delta E^2_{20} + \delta E^3_{20} \tag{11}$$

where

$$\delta E^1_{20} = \int_0^{r_0-a} R^2_{20}(\frac{-e^2}{r_0})r^2\,dr = \frac{-e^2}{(2a_0)^3\,r_0}\int_0^{r_0-a}(4r^2-\frac{8r^3}{a_0})dr =$$

$$\frac{-e^2}{(2a_0)^3}\left[r_0^2(\frac{4}{3}-\frac{2r_0}{a_0}) + r_0a\big(-4+\frac{8r_0}{a_0}\big) + a^2\big(4-\frac{12r_0}{a_0}\big)\right], \quad (12)$$

$$\delta E^2_{20} = \int_0^{r_0+a} R^2_{20}(\frac{+e^2}{r})r^2\,dr = \frac{+e^2}{(2a_0)^3}\int_0^{r_0+a}(4r-\frac{8r^2}{a_0})dr =$$

$$\frac{e^2}{(2a_0)^3}\left[r_0^2\big(2-\frac{8r_0}{3a_0}\big) + r_0a\big(4-\frac{8r_0}{a_0}\big) + a^2\big(2-\frac{8r_0}{a_0}\big)\right], \quad (13)$$

and

$$\delta E^3_{20} = \int_{r_0-a}^{r_0+a} R^2_{20}\,V_b\,r^2\,dr\,. \quad (14)$$

δE^3_{20} will be approximated using the mean value theorem for integrals. Replace r by the mean value of r, which is r_0, and replace V_b by its mean value, which is $-e^2/(r_0+a/2)$. Then

$$\delta E^3_{20} = \int_{r_0-a}^{r_0+a} R^2_{20}V_b\,r^2\,dr \approx \frac{2a}{(2a_0)^3}\big(4r_0^2-\frac{8r_0^3}{a_0}\big)\frac{-e^2}{r_0+a/2}$$

$$\approx \frac{-e^2}{(2a_0)^3}\big(4r_0-\frac{8r_0^2}{a_0}\big)(1-a/2r_0) = \frac{e^2}{(2a_0)^3}\Big[-8r_0a\big(1-\frac{2r_0}{a_0}\big)+4a^2\big(1-\frac{2r_0}{a_0}\big)\Big]. \quad (15)$$

The result is

$$\delta E_{20} = \frac{e^2}{(2a_0)^3}\Big[\frac{2}{3}r_0^2\big(1-\frac{r_0}{a_0}\big) + 2a^2\big(1-2\frac{r_0}{a_0}\big)\Big]. \quad (16)$$

This calculation can also be applied to muonic hydrogen.

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References

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