EXTENDED QUARK EFFECT ON HYDROGEN SPECTRUM(1S)

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1 A QUARK MODEL OF THE PROTON

The purpose of this paper is to test a theory of how the quarks in the proton might affect the H-atom spectrum. There are three quarks making up the proton. There are two positively charged quarks, which are called up quarks, each one having a charge $q_+ = +2e/3$ where $-e$ is the electron charge. There is one negatively charged quark, which is called a down quark, and has a charge $q_- = -e/3$.

The quark charge is assumed to be uniformly distributed on a spherical shell of radius r_o. Thus the charge probability density of each quark is assumed to be $q_\pm \delta(y - r_o)/(4\pi r_0^2)$ where y is the distance from the quark center to a spherical shell of radius r_o. Take the proton to be at the origin, and the proton radius to also be r_o. When only the proton is present, take the three quark centers to also be at the origin.

For a proton in the hydrogen atom, it will be assumed that the electron attracts the positively charged quarks, and repels the negatively charged quark. When the electron is a distance r from the origin, assume the up quark is displaced by a distance δ_+. As the electron moves closer to the origin, it is expected that δ_+ would increase. In the interest of mathematical simplicity, take δ_+ to be constant. So a positive quark is always assumed to be displaced from the origin on the line from the origin to the electron by a constant distance δ_+ toward the electron. Similarly, the negative quark is always assumed to be displaced from the origin on the extension of the line from the electron through the origin by a constant distance δ_-. The displacements δ_+ and δ_- will be assumed to be much smaller than both a_0, the Bohr radius and r_o, so $\delta_\pm << r_o << a_0$.

Often the Taylor expansion $\exp(-2\delta_\pm/a_0) = 1 - 2\delta_\pm/a_0 + 4\delta_\pm^2/(2a_0^2) + \cdots$ will enter the equations for the energy shifts. Only terms in the energy shifts like $eq_\pm/a_0$,

$eq_\pm r_o^2/a_0^3$ and $eq_\pm \delta_\pm/a_0^2$ will be kept. Terms such as $eq_\pm r_o \delta_\pm^n/a_0^{n+2}$ for $n \geq 1$ and $eq_\pm \delta_\pm^n/a_0^{n+1}$ for $n \geq 2$ are negligible in comparison, and will be discarded.

2 POTENTIAL ENERGY OF AN UP QUARK AND A POINT ELECTRON

Now consider an up quark displaced a distance δ_+ as in the model. Let $\mathbf{r}_+$ be the vector from the center of the quark to the electron, and let r_+ be the corresponding distance. Let $\mathbf{y}$ be the vector from the quark center to an element of quark charge, and let y be the corresponding distance. Define $\mathbf{r}'$ to be the vector from an element of quark charge to the electron, and r' is the corresponding distance. The three vectors form a triangle, and $r'^2 = r_+^2 + y^2 - 2r_+ y \cos(\theta)$ where θ is the angle between $\mathbf{r}_+$ and $\mathbf{y}$. The differential potential of the quark is

$$d\Phi_+ = \frac{q_+}{r'} \frac{\delta(y - r_o)}{4\pi r_o^2} y^2 \, \sin(\theta) \, d\theta \, d\phi \, dy \tag{1}$$

Set $u = -\cos(\theta)$. Then

$$\Phi_+ = q_+ \int_{-1}^{+1} \frac{du}{2\sqrt{r_+^2 + r_o^2 + 2r_+ r_o \, u}} = \frac{q_+}{2r_+ r_o} \sqrt{r_+^2 + r_o^2 + 2r_+ r_o \, u} \, \Big|_{-1}^{+1} =$$
$$\frac{q_+}{2r_+ r_o}[r_+ + r_o - (r_+ - r_o)] H(r_+ - r_o) + \frac{q_+}{2r_+ r_o}[r_+ + r_o - (r_o - r_+)] H(r_o - r_+) =$$
$$\frac{q_+}{r_+} H(r_+ - r_o) + \frac{q_+}{r_o} H(r_o - r_+) \tag{2}$$

where $H()$ is the unit step function. Note that the square root represents a distance , so it must be positive.

Let r denote the distance from the origin to the electron. Then $r = r_+ + \delta_+$, and the potential energy of a top quark and the electron is conveniently written

$$V_+ = -\frac{eq_+}{r_+} H(r_+ - r_o) - \frac{eq_+}{r_o} H(r_o + \delta_+ - r) \, . \tag{3}$$

3 POTENTIAL ENERGY OF A DOWN QUARK AND THE ELECTRON

Now consider the down quark displaced a distance δ_- as in the model. Let $\mathbf{r}_-$ be the vector from the center of the quark to the electron, and let r_- be the corresponding distance. Again, let $\mathbf{r}'$ be the vector from an element of quark charge to the electron. Then $r'^2 = r_-^2 + y^2 - 2r_- y\cos(\theta)$ where θ is the angle between $\mathbf{r}_-$ and $\mathbf{y}$. The differential potential of the quark is

$$d\Phi_- = \frac{q_-}{r'}\frac{\delta(y - r_o)}{4\pi r_o^2}y^2\,\sin(\theta)\,d\theta\,d\phi\,dy \tag{4}$$

Set $u = -\cos(\theta)$. Then

$$\Phi_- = q_-\int_{-1}^{+1}\frac{du}{2\sqrt{r_-^2 + r_o^2 + 2r_- r_o\,u}} = \frac{q_-}{r_-}H(r_- - r_o) + \frac{q_-}{r_o}H(r_o - r_-)\,. \tag{5}$$

Since $r_- = r + \delta_-$, the potential energy of the down quark and the electron is conveniently written

$$V_- = -\frac{eq_-}{r_-}H(r_- - r_o) - \frac{eq_-}{r_o}H(r_o - \delta_- - r)\,. \tag{6}$$

4 THE HAMILTONIAN

For the proton and quarks at the origin, the unperturbed Hamiltonian of an up quark and the electron is $H_{o+} = p^2/2\mu_+ - eq_+/r$ where μ_+ is the reduced mass of an up quark and the electron. The Hamiltonian of an up quark displaced by the distance δ_+ and an electron is

$$H_+ = \frac{p^2}{2\mu_+} + V_+ = \frac{p^2}{2\mu_+} - \frac{eq_+}{r} + \frac{eq_+}{r} + V_+ = H_{o+} + H'_+ \tag{7}$$

where the perturbing Hamiltonian for an up quark and the electron is

$$H'_+ = +\frac{eq_+}{r} - \frac{eq_+}{r_+}H(r_+ - r_o) - \frac{eq_+}{r_o}H(r_o + \delta_+ - r) \tag{8}$$

The unperturbed Hamiltonian of the down quark and the electron is given by $H_{o-} = p^2/2\mu_- - eq_-/r$ where μ_- is the reduced mass of the down quark and the electron. The Hamiltonian for the down quark displaced by the distance δ_- and an electron is

$$H_- = \frac{p^2}{2\mu_-} + V_- = \frac{p^2}{2\mu_-} - \frac{eq_-}{r} + \frac{eq_-}{r} + V_- = H_{o-} + H'_- \tag{9}$$

where the perturbing Hamiltonian for the down quark and the electron is

$$H'_- = \frac{eq_-}{r} - \frac{eq_-}{r_-}H(r_- - r_o) - \frac{eq_-}{r_0}H(r_o - \delta_- - r)\,. \tag{10}$$

So the perturbing Hamiltonian for the quarks and the electron is $H' = 2H'_+ + H'_-$.

5 ENERGY SHIFT DUE TO H'_+

The energy shift associated with H'_+ is

$$\delta E_+ = \int \psi^*(r) H'_+ \psi(r)\, d^3r = I_{1+} + I_{2+} + I_{3+} \qquad (11)$$

where $d^3r = r^2\, \sin(\theta)\, d\theta\, d\phi\, dr$, the unperturbed wave function for the $1S$ energy level is $\psi(r) = 2\exp(-r/a_0)/\sqrt{4\pi a_0^3}$,

$$I_{1+} = \int \frac{4\exp(-2r/a_0)}{a_0^3 4\pi} \Big(\frac{eq_+}{r}\Big) r^2\, \sin(\theta)\, d\theta\, d\phi\, dr = \frac{eq_+}{a_0}\,, \qquad (12)$$

$$I_{2+} = -\int \frac{4\exp(-2r/a_0)}{a_0^3} \Big(\frac{eq_+}{r_+} H(r_+ - r_o)\Big)\, r^2\, dr\,, \text{ and} \qquad (13)$$

$$I_{3+} = -\int \frac{4\exp(-2r/a_0)}{a_0^3} \Big(\frac{eq_+}{r_o} H(r_o + \delta_+ - r)\Big)\, r^2\, dr\,. \qquad (14)$$

6 CALCULATION OF I_{2+}

Substitute $r = r_+ + \delta_+$ in Eq. (13). Then

$$I_{2+} = -\frac{4eq_+}{a_0^3} \exp(-2\delta_+/a_0) \int_{r_o}^{\infty} \exp(-2r_+/a_0) \Big(r_+ + 2\delta_+ + \frac{\delta_+^2}{r_+}\Big) dr_+\,. \quad (15)$$

I_{2+} is the sum of the following integrals:

$$\begin{aligned} I_{2+1} = -\frac{4eq_+}{a_0^3} \exp(-2\delta_+/a_0) \int_{r_o}^{\infty} r_+ \exp(-2r_+/a_0) dr_+ = \\ -\frac{4eq_+}{a_0^3} \exp(-2\delta_+/a_0) \exp(-2r_+/a_0) \Big(\frac{-r_+ a_0}{2} - \frac{a_0^2}{4}\Big)\Big|_{r_o}^{\infty} = \\ -4\frac{eq_+}{a_0^3}\Big(1 - 2\frac{r_o + \delta_+}{a_0} + 4\frac{(r_o+\delta_+)^2}{2a_0^2}\Big)\Big(\frac{r_o a_0}{2} + \frac{a_0^2}{4}\Big) = \\ -\frac{eq_+}{a_0} + \frac{2eq_+ r_o^2}{a_0^3} + 2\frac{eq_+\delta_+}{a_0^2}\,, \end{aligned} \quad (16)$$

$$\begin{aligned} I_{2+2} = -\frac{4eq_+}{a_0^3} \exp(-2\delta_+/a_0) 2\delta_+ \int_{r_o}^{\infty} \exp(-2r_+/a_0) dr_+ = \\ -\frac{4eq_+}{a_0^3} \exp(-2\delta_+/a_0) 2\delta_+ \exp(-2r_+/a_0) \Big(\frac{-a_0}{2}\Big)\Big|_{r_o}^{\infty} = -4\frac{eq_+}{a_0^2}\delta_+\,, \text{ and} \end{aligned} \quad (17)$$

$$I_{2+3} = -\frac{4eq_+}{a_0^3}\exp(-2\delta_+/a_0)\delta_+^2\int_{r_o}^{\infty}\exp(-2r_+/a_0)\frac{dr_+}{r_+}\,. \qquad (18)$$

The integral in I_{2+3} converges, but I_{2+3} contains δ_+^2, so it will be discarded.

Add the results together, and find $I_{2+} = -eq_+/a_0 + 2eq_+r_o^2/a_0^3 - 2eq_+\delta_+/a_0^2\,.$

7 CALCULATION OF I_{3+}

By Eq. (14)

$$I_{3+} = -\frac{4eq_+}{a_0^3}\int_0^{r_o+\delta_+}\exp(-2r/a_0)r^2\frac{dr}{r_o}\,. \qquad (19)$$

Approximate the exponential by 1. Then

$$I_{3+} = -\frac{4eq_+}{a_0^3}\int_0^{r_o+\delta_+}r^2\frac{dr}{r_o} = -\frac{4eq_+}{3a_0^3}r_o^2\,. \qquad (20)$$

Add the contributions to the energy shift from an up quark and the electron, and find $\delta E_+ = 2eq_+r_o^2/(3a_0^3) - 2eq_+\delta_+/a_0^2\,.$

8 ENERGY SHIFT DUE TO H'_-

The energy shift associated with H'_- is

$$\delta E_- = \int \psi^*(r)H'_-\psi(r)\,d^3r = I_{1-} + I_{2-} + I_{3-} \qquad (21)$$

where

$$I_{1-} = \int \frac{4\exp(-2r/a_0)}{a_0^3 4\pi}\Big(\frac{eq_-}{r}\Big)r^2\,\sin(\theta)\,d\theta\,d\phi\,dr = \frac{eq_-}{a_0}\,, \qquad (22)$$

$$I_{2-} = -\int \frac{4\exp(-2r/a_0)}{a_0^3}\Big(\frac{eq_-}{r_-}H(r_- - r_o)\Big)\,r^2\,dr\,,\ \text{and} \qquad (23)$$

$$I_{3-} = -\int \frac{4\exp(-2r/a_0)}{a_0^3}\Big(\frac{eq_-}{r_o}H(r_o - \delta_+ - r)\Big)\,r^2\,dr\,. \qquad (24)$$

9 CALCULATION OF I_{2-}

Substitute $r = r_- - \delta_-$ in Eq. (23). Then

$$I_{2-} = -\frac{4eq_-}{a_0^3}\exp(+2\delta_-/a_0)\int_{r_o}^{\infty}\exp(-2r_-/a_0)\Big(r_- - 2\delta_- + \frac{\delta_-^2}{r_-}\Big)dr_-\,. \qquad (25)$$

I_{2-} is the sum of the following integrals:

$$I_{2-1} = -\frac{4eq_-}{a_0^3}\exp(+2\delta_-/a_0)\int_{r_o}^{\infty} r_- \exp(-2r_-/a_0)dr_- =$$

$$-\frac{4eq_-}{a_0^3}\exp\big(-2(r_o-\delta_-)/a_0\big)\Big(\frac{r_o a_0}{2}+\frac{a_0^2}{4}\Big) =$$

$$-\frac{4eq_-}{a_0^3}\Big(1-2\frac{(r_o-\delta_-)}{a_0}+2\frac{(r_o+\delta_-)^2}{a_0^2}\Big)\Big(\frac{r_o a_0}{2}+\frac{a_0^2}{4}\Big) = -\frac{eq_-}{a_0}+\frac{eq_- 2r_o^2}{a_0^3}-\frac{2eq_-\delta_-}{a_0^2}\,, \tag{26}$$

$$I_{2-2} = -\frac{4eq_-}{a_0^3}\exp(+2\delta_-/a_0)(-2\delta_-)\int_{r_o}^{\infty} \exp(-2r_-/a_0)dr_- =$$

$$-\frac{4eq_-}{a_0^3}\exp\big(-2(r_o-\delta_-)/a_0\big)(-\delta_- a_0) =$$

$$-\frac{4eq_-}{a_0^3}\Big(1-2\frac{(r_o-\delta_-)}{a_0}\Big)(-\delta_- a_0) = \frac{4eq_-\delta_-}{a_0^2}\,, \text{and} \tag{27}$$

$$I_{2-3} = -\frac{4eq_-}{a_0^3}\exp(+2\delta_-/a_0)(-2\delta_-)\delta_-^2\int_{r_o}^{\infty} \exp(-2r_-/a_0)\frac{dr_-}{r_-}\,. \tag{28}$$

The integral in I_{2-3} converges, but I_{2-3} is multiplied by δ_-^2, so I_{2-3} is set equal to zero. Thus $I_{2-} = -eq_-/a_0 + 2eq_- r_o^2/a_0^3 + 2eq_-\delta_-/a_0^2$.

10 CALCULATION OF I_{3-}

$$I_{3-} = -\frac{4eq_-}{a_0^3}\int_0^{r_o-\delta_-}\exp(-2r/a_0)r^2\,\frac{dr}{r_o} = -\frac{4eq_-}{a_0^3}\int_0^{r_o-\delta_-} r^2\,\frac{dr}{r_o} =$$

$$-\frac{4eq_-}{a_0^3}\frac{r^3}{3r_o}\Big|_0^{r_o-\delta_-} = -\frac{4eq_- r_o^2}{3a_0^3}\,. \tag{29}$$

Add the contributions to the energy shift from the down quark and the electron, and find $\delta E_- = 2eq_- r_o^2/(3a_0^3) + 2eq_-\delta_-/a_0^2$.

11 TOTAL ENERGY SHIFT

The total energy shift of the $1S$ level is the sum of $2\delta E_+$ and δE_-, so

$$\delta E(1S) = 2\Big(\frac{2eq_+ r_o^2}{3a_0^3}-\frac{2eq_+\delta_+}{a_0^2}\Big)+\frac{eq_- r_o^2}{3a_0^3}+\frac{2eq_-\delta_-}{a_0^2} \tag{30}$$

Let m_+ be the mass of an up quark, and let m_- be the mass of the down quark. Take $m_- = 2m_+$(this is approximate). With the center of mass of the quarks at the origin, $2m_+\delta_+ = m_-\delta_-$, so $\delta_- = \delta_+$. Substitute $q_+ = 2e/3$ and $q_- = -e/3$ into Eq. (30), and find

$$\delta E(1S) = +\frac{8e^2r_o^2}{9a_0^3} - \frac{2e^2r_o^2}{9a_0^3} - \frac{8e^2\delta_+}{3a_0^2} - \frac{2e^2\delta_+}{3a_0^2} = +\frac{2e^2r_o^2}{3a_0^2} - \frac{10}{3}\frac{e^2\delta_+}{a_0^2}\,. \quad (31)$$

For the the quark charge density $\rho = q\delta(y - r_o)/(4\pi r_o^2)$, the mean square radius of the quark is r_o^2

The energy shift due to proton size is known to be $\delta E = 2\pi e^2 r_p^2|\psi(r = 0)|^2/3$ where r_p^2 is the mean square radius of the proton.[1] For the $1S$ level, $\psi(r) = 2\exp(-r/a_0)/\sqrt{a_0^3 4\pi}$, so

$$\delta E(1S) = \frac{2}{3a_0^3}e^2r_p^2. \quad (32)$$

Since r_o^2 is the mean square radius of a quark, the first term in Eq. (31) agrees with the accepted proton size term. The second term due to polarization of the quarks when applied to the $2S$ and $2P$ states might explain the proton size discrepancy found in muonic hydrogen data.[2]

In future papers, the effect on the hydrogen atom spectrum of both quark displacement as in the model and proton size will be studied.

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References

[1] http://www.fuw.edu.pl/ krp/papers/pohl.pdf

[2] http://www.electronformfactor.com/Quark Effect On Muonic Hydrogen Spectrum.