# PROTON ENERGY LEVELS BASED ON A QUARK MODEL

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## 1 A QUARK MODEL OF THE PROTON

The purpose of this paper is to test a theory of how the quarks in the proton might affect the proton spectrum. There are three quarks making up the proton. The model takes the quarks to be point particles. There are two positively charged quarks, which are called up quarks, each one having a charge $q_+ = +2e/3$ where $+e$ is the proton charge. There is one negatively charged quark, which is called a down quark, and has a charge $q_- = -e/3$. The proton model will be simplified in order to focus on the proton energy levels. For example, take the three quark masses to be the same, and label that mass m.

## 2 KINETIC ENERGY OF THE QUARKS

The position vectors of the three quarks are labeled $\mathbf{r}_1, \mathbf{r}_2$, and $\mathbf{r}_3$. Then the kinetic energy of the three quarks is

$$T = m(\dot{\mathbf{r}}_1^2 + \dot{\mathbf{r}}_2^2 + \dot{\mathbf{r}}_3^2)/2\,. \tag{1}$$

Introduce the center of mass coordinate $\mathbf{R}$ by

$$3m\mathbf{R} = m(\mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3) \tag{2}$$

Introduce the relative coordinates $\mathbf{r}_{21} = \mathbf{r}_2 - \mathbf{r}_1$, $\mathbf{r}_{32} = \mathbf{r}_3 - \mathbf{r}_2$, and $\mathbf{r}_{13} = \mathbf{r}_1 - \mathbf{r}_3$. Substitute $\mathbf{r}_2 = \mathbf{r}_{21} + \mathbf{r}_1$, and $\mathbf{r}_3 = \mathbf{r}_1 - \mathbf{r}_{13}$ into Eq. 2, and find

$$\mathbf{r}_1 = \mathbf{R} + \frac{\mathbf{r}_{13}}{3} - \frac{\mathbf{r}_{21}}{3}\,. \tag{3}$$

Similarly find

$$\mathbf{r}_2 = \mathbf{R} + \frac{\mathbf{r}_{21}}{3} - \frac{\mathbf{r}_{32}}{3}, \text{ and} \tag{4}$$

$$\mathbf{r}_3 = \mathbf{R} + \frac{\mathbf{r}_{32}}{3} - \frac{\mathbf{r}_{13}}{3}\,. \tag{5}$$

Take the time derivative of equation (3), square, and find

$$\dot{\mathbf{r}}_\mathbf{1}^\mathbf{2} = \dot{\mathbf{R}}^\mathbf{2} + \frac{2\dot{\mathbf{R}}\cdot\dot{\mathbf{r}}_\mathbf{13}}{3} - \frac{2\dot{\mathbf{R}}\cdot\dot{\mathbf{r}}_\mathbf{21}}{3} - \frac{2\dot{\mathbf{r}}_\mathbf{13}\cdot\dot{\mathbf{r}}_\mathbf{21}}{9} + \frac{\dot{\mathbf{r}}_\mathbf{13}^2}{9} + \frac{\dot{\mathbf{r}}_\mathbf{21}^2}{9}\,. \tag{6}$$

Similarly

$$\dot{\mathbf{r}}_\mathbf{2}^\mathbf{2} = \dot{\mathbf{R}}^\mathbf{2} + \frac{2\dot{\mathbf{R}}\cdot\dot{\mathbf{r}}_\mathbf{21}}{3} - \frac{2\dot{\mathbf{R}}\cdot\dot{\mathbf{r}}_\mathbf{32}}{3} - \frac{2\dot{\mathbf{r}}_\mathbf{21}\cdot\dot{\mathbf{r}}_\mathbf{32}}{9} + \frac{\dot{\mathbf{r}}_\mathbf{21}^2}{9} + \frac{\dot{\mathbf{r}}_\mathbf{32}^2}{9} \text{ and} \tag{7}$$

$$\dot{\mathbf{r}}_\mathbf{3}^\mathbf{2} = \dot{\mathbf{R}}^\mathbf{2} + \frac{2\dot{\mathbf{R}}\cdot\dot{\mathbf{r}}_\mathbf{32}}{3} - \frac{2\dot{\mathbf{R}}\cdot\dot{\mathbf{r}}_\mathbf{13}}{3} - \frac{2\dot{\mathbf{r}}_\mathbf{32}\cdot\dot{\mathbf{r}}_\mathbf{13}}{9} + \frac{\dot{\mathbf{r}}_\mathbf{32}^2}{9} + \frac{\dot{\mathbf{r}}_\mathbf{13}^2}{9}\,. \tag{8}$$

Substitute the above results into the equation for the kinetic energy and get

$$T = \frac{3m\dot{\mathbf{R}}^\mathbf{2}}{2} + \frac{m}{9}\left[(\dot{\mathbf{r}}_\mathbf{13} - \dot{\mathbf{r}}_\mathbf{21})^2 + (\dot{\mathbf{r}}_\mathbf{32} - \dot{\mathbf{r}}_\mathbf{13})^2 + (\dot{\mathbf{r}}_\mathbf{21} - \dot{\mathbf{r}}_\mathbf{32})^2\right] \tag{9}$$

. The kinetic energy can be put in the form

$$T = \frac{3m\dot{\mathbf{R}}^\mathbf{2}}{2} + \frac{m}{9}\left[(2\dot{\mathbf{r}}_\mathbf{1} - \dot{\mathbf{r}}_\mathbf{2} - \dot{\mathbf{r}}_\mathbf{3})^\mathbf{2} + (\mathbf{2}\dot{\mathbf{r}}_\mathbf{2} - \dot{\mathbf{r}}_\mathbf{3} - \dot{\mathbf{r}}_\mathbf{1})^2 + (2\dot{\mathbf{r}}_\mathbf{3} - \dot{\mathbf{r}}_\mathbf{2} - \dot{\mathbf{r}}_\mathbf{1})^2\right]. \tag{10}$$

# 3 JACOBI COORDINATES[1]

Note that

$$(2\dot{\mathbf{r}}_\mathbf{1} - \dot{\mathbf{r}}_\mathbf{2} - \dot{\mathbf{r}}_\mathbf{3})^\mathbf{2} + (\mathbf{2}\dot{\mathbf{r}}_\mathbf{2} - \dot{\mathbf{r}}_\mathbf{3} - \dot{\mathbf{r}}_\mathbf{1})^2 = \frac{9}{2}(\dot{\mathbf{r}}_\mathbf{2} - \dot{\mathbf{r}}_\mathbf{1})^2 + \frac{1}{2}(2\dot{\mathbf{r}}_\mathbf{3} - \dot{\mathbf{r}}_\mathbf{2} - \dot{\mathbf{r}}_\mathbf{1})^2 \tag{11}$$

Add $(2\dot{\mathbf{r}}_3 - \dot{\mathbf{r}}_2 - \dot{\mathbf{r}}_1)^2$ to Eq. 11. Then

$$T = \frac{3m\dot{\mathbf{R}}^\mathbf{2}}{2} + \frac{m\dot{\mathbf{r}}_\mathbf{21}^2}{2} + \frac{m\dot{\mathbf{q}}_\mathbf{3}^2}{2} \tag{12}$$

where $\mathbf{q}_3 = (2\mathbf{r}_3 - \mathbf{r}_2 - \mathbf{r}_1)\sqrt{3}$. So $\mathbf{r}_1, \mathbf{r}_2$, and $\mathbf{r}_3$ are replaced by the Jacobi coordinates $\mathbf{R}, \mathbf{r}_{21}$, and $\mathbf{q}_3$ .

By a similar calculation, the kinetic energy can also be written

$$T = \frac{3m\dot{\mathbf{R}}^\mathbf{2}}{2} + \frac{m\dot{\mathbf{r}}_\mathbf{32}^2}{2} + \frac{m\dot{\mathbf{q}}_\mathbf{1}^2}{2} \tag{13}$$

where $\mathbf{q}_1 = (2\mathbf{r}_1 - \mathbf{r}_3 - \mathbf{r}_2)\sqrt{3}$, and $\mathbf{R}, \mathbf{r}_{32}$, and $\mathbf{q}_1$ are another set of Jacobi coordinates. Finally the kinetic energy can also be written

$$T = \frac{3m\dot{\mathbf{R}}^{\mathbf{2}}}{2} + \frac{m\dot{\mathbf{r}}_{\mathbf{13}}^2}{2} + \frac{m\dot{\mathbf{q}}_{\mathbf{2}}^2}{2} \tag{14}$$

where $\mathbf{q}_2 = (2\mathbf{r}_2 - \mathbf{r}_3 - \mathbf{r}_1)/\sqrt{3}$, and $\mathbf{R}, \mathbf{r}_{13}$, and $\mathbf{q}_2$ are our final set of Jacobi coordinates.

When there are no external forces acting on the proton, the center of mass moves with a constant velocity. Our interest is in the internal energy of the proton, so set $\dot{\mathbf{R}} = 0$, and take the center of mass to be at the origin. The kinetic energy can then be written in the following three ways:

$$T = \frac{m}{2}\left(\dot{\mathbf{r}}_{\mathbf{32}}^{\mathbf{2}} + \dot{\mathbf{q}}_{\mathbf{1}}^2\right) = \frac{m}{2}\left(\dot{\mathbf{r}}_{\mathbf{13}}^{\mathbf{2}} + \dot{\mathbf{q}}_{\mathbf{2}}^2\right) = \frac{m}{2}\left(\dot{\mathbf{r}}_{\mathbf{21}}^{\mathbf{2}} + \dot{\mathbf{q}}_{\mathbf{3}}^2\right). \tag{15}$$

By the introduction of Jacobi coordinates, the kinetic energy is put in a useful form. It is convenient to write the kinetic energy as

$$T = \frac{1}{3}\left(\frac{m\dot{\mathbf{r}}_{\mathbf{21}}^2}{2} + \frac{m\dot{\mathbf{q}}_{\mathbf{3}}^2}{2} + \frac{m\dot{\mathbf{r}}_{\mathbf{32}}^2}{2} + \frac{m\dot{\mathbf{q}}_{\mathbf{1}}^2}{2} + \frac{m\dot{\mathbf{r}}_{\mathbf{13}}^2}{2} + \frac{m\dot{\mathbf{q}}_{\mathbf{2}}^2}{2}\right). \tag{16}$$

# 4 NEVER MAKE A CALCULATION UNTIL YOU KNOW THE ANSWER[2]

The answer is that the quarks must be confined, and there must be a relatively large energy term which is approximately equal to the $m_p c^2$, the proton rest energy. Guess that a potential energy proportional to the sum of squares of the $q$ s will satisfy the above conditions. Choose the proportionality constant to be $k/6$. The answer is that there must be a set of discrete energy levels that explain the $\gamma$ radiation of an excited nucleus by means of a pair of quarks transitioning from a higher energy state to a lower energy state. Guess that this potential energy is proportional to minus the reciprocal distance between pairs of quarks. Choose the proportionality constant to be $-\lambda/3$. Then

$$V = -\frac{\lambda}{3|\mathbf{r}_{21}|} - \frac{\lambda}{3|\mathbf{r}_{32}|} - \frac{\lambda}{3|\mathbf{r}_{13}|} + \frac{k}{6}\left(\mathbf{q}_1^2 + \mathbf{q}_2^2 + \mathbf{q}_3^2\right). \tag{17}$$

At this point, $\lambda$ and $k$ are unknown. The terms in the above potential energy have been suggested in reference (1). Then

$$\begin{aligned} T + V = &+\frac{m}{6}\dot{\mathbf{r}}_{\mathbf{21}}^2 - \frac{\lambda}{3|\mathbf{r}_{21}|} + \frac{m}{6}\dot{\mathbf{r}}_{\mathbf{32}}^2 - \frac{\lambda}{3|\mathbf{r}_{32}|} + \frac{m}{6}\dot{\mathbf{r}}_{\mathbf{13}}^2 - \frac{\lambda}{3|\mathbf{r}_{13}|} \\ &+ \frac{m}{6}\dot{\mathbf{q}}_{\mathbf{1}}^{\mathbf{2}} + \frac{k}{6}\mathbf{q}_1^2 + + \frac{m}{6}\dot{\mathbf{q}}_{\mathbf{2}}^{\mathbf{2}} + \frac{k}{6}\mathbf{q}_2^2 + \frac{m}{6}\dot{\mathbf{q}}_{\mathbf{3}}^{\mathbf{2}} + \frac{k}{6}\mathbf{q}_3^2. \end{aligned} \tag{18}$$

# 5 THE CALCULATION

Use Eq. 18 to write down the Schrodinger equation.

$$\Big[\frac{-\hbar^2}{6m}\nabla^2_{\mathbf{r}21} - \frac{\lambda}{3|\mathbf{r}_{21}|} + \frac{-\hbar^2}{6m}\nabla^2_{\mathbf{r}32} - \frac{\lambda}{3|\mathbf{r}_{32}|} + \frac{-\hbar^2}{6m}\nabla^2_{\mathbf{r}13} - \frac{\lambda}{3|\mathbf{r}_{13}|} - \frac{\hbar^2}{6m}\nabla^2_{q1} + \frac{k}{6}q_1^2 - \frac{\hbar^2}{6m}\nabla^2_{q2} + \frac{k}{6}q_2^2 - \frac{\hbar^2}{6m}\nabla^2_{q3} + \frac{k}{6}q_3^2\Big]\psi = E\psi\,. \quad (19)$$

Eq. 19 separates into three 3-dimensional harmonic oscillator Schrodinger equations and three hydrogen like Schrodinger equations. The ground state solution to $1/3\big(-\hbar^2\nabla^2_{q1}/(2m) + kq_1^2/2\big)\psi(q_1) = E_{q_1}\psi(q_1)$ is $\psi(q_1) = (m\omega/\hbar\pi)^{3/4}\exp(-\alpha^2 q_1^2/2)$ and $E_{q_1} = \hbar\omega/2$ where $\omega = \sqrt{k/m}$ and $\alpha^2 = m\omega/\hbar$. The ground state solution to $1/3\big(-\hbar^2\nabla^2_{r21}/(2m) - \lambda/|\mathbf{r}_{21}|\big)\psi(|\mathbf{r}_{21}|) = E_{r_{21}}\psi(|\mathbf{r}_{21}|)$ is $\psi(|\mathbf{r}_{21}|) = 2\exp(-|\mathbf{r}_{21}|/a_\lambda)/(4\pi a_\lambda^3)^{1/2}$ and $E_{r21} = -\lambda/(6a_\lambda)$ where $a_\lambda = \hbar^2/m\lambda$. The total ground state energy of the three quarks is given by $3\hbar\omega/2 - \lambda/2a_\lambda$. As part of the model, take $\lambda/a_\lambda << \hbar\omega$. Then $3\hbar\omega/2 \approx m_p c^2$ gives an estimation of $\lambda$ and $k$.

The discrete energy associated with a pair of quarks is $-\lambda/(6n^2 a_\lambda)$ where $n$ is a positive integer. As an example, the energy change in a transition from the $n = 2$ level to the $n = 1$ level is $-\lambda/(8a_\lambda)$. If this energy change results in the emission of a photon, then a measurement of the photon energy yields an estimation of $\lambda$.

The theory presented is far too simple to be conclusive. It is hoped that the idea presented will provide a starting point in the study of the proton spectrum.

# ACKNOWLEDGMENTS

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# References

[1] Jean-Marc Richard *An introduction to the quark model*

[2] Edwin Taylor and John Wheeler *Spacetime Physics* (Freeman 1966) p 60.